1

Prerequisites

Figure 1 Credit: Andreas Kambanls

CHAPTER OUTLINE

Introduction

It's a cold day in Antarctica. In fact, it's always a cold day in Antarctica. Earth's southernmost continent, Antarctica experiences the coldest, driest, and windiest conditions known. The coldest temperature ever recorded, over one hundred degrees below zero on the Celsius scale, was recorded by remote satellite. It is no surprise then, that no native human population can survive the harsh conditions. Only explorers and scientists brave the environment for any length of time.

Measuring and recording the characteristics of weather conditions in Antarctica requires a use of different kinds of numbers. Calculating with them and using them to make predictions requires an understanding of relationships among numbers. In this chapter, we will review sets of numbers and properties of operations used to manipulate numbers. This understanding will serve as prerequisite knowledge throughout our study of algebra and trigonometry.

LEARNING OBJECTIVES

In this section students will:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

1.1 REAL NUMBERS: ALGEBRA ESSENTIALS

It is often said that mathematics is the language of science. If this is true, then the language of mathematics is numbers. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity—a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing? From earliest times, people had thought of a "base state" while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly, there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

Because of the evolution of the number system, we can now perform complex calculations using these and other categories of real numbers. In this section, we will explore sets of numbers, calculations with different kinds of numbers, and the use of numbers in expressions.

Classifying a Real Number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as $\{1, 2, 3, \ldots\}$ where the ellipsis $(\ldots)$ indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \ldots\}$.

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

negative integers	zero	positive integers
$\ldots, -3, -2, -1,$	$0,$	$1, 2, 3, \ldots$

The set of **rational numbers** is written as $\left\{ \dfrac{m}{n} \,\middle|\, m \text{ and } n \text{ are integers and } n \neq 0 \right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\dfrac{15}{8} = 1.875$, or
2. a repeating decimal: $\dfrac{4}{11} = 0.36363636\ldots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

Example 1 Writing Integers as Rational Numbers

Write each of the following as a rational number.

a. 7 **b.** 0 **c.** -8

Solution Write a fraction with the integer in the numerator and 1 in the denominator.

a. $7 = \dfrac{7}{1}$ **b.** $0 = \dfrac{0}{1}$ **c.** $-8 = -\dfrac{8}{1}$

Try It #1

Write each of the following as a rational number.

a. 11 **b.** 3 **c.** -4

Example 2 Identifying Rational Numbers

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $-\dfrac{5}{7}$ **b.** $\dfrac{15}{5}$ **c.** $\dfrac{13}{25}$

Solution Write each fraction as a decimal by dividing the numerator by the denominator.

a. $-\dfrac{5}{7} = -0.\overline{714285}$, a repeating decimal

b. $\dfrac{15}{5} = 3$ (or 3.0), a terminating decimal

c. $\dfrac{13}{25} = 0.52$, a terminating decimal

Try It #2

Write each of the following rational numbers as either a terminating or repeating decimal.

a. $\dfrac{68}{17}$ **b.** $\dfrac{8}{13}$ **c.** $-\dfrac{17}{20}$

Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\dfrac{3}{2}$, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

$$\{h \mid h \text{ is not a rational number}\}$$

Example 3 Differentiating Rational and Irrational Numbers

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\sqrt{25}$ **b.** $\dfrac{33}{9}$ c. $\sqrt{11}$ **d.** $\dfrac{17}{34}$ e. 0.3033033303333…

Solution

a. $\sqrt{25}$: This can be simplified as $\sqrt{25} = 5$. Therefore, $\sqrt{25}$ is rational.

b. $\frac{33}{9}$: Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$\frac{33}{9}=\frac{\overset{11}{\cancel{33}}}{\underset{3}{\cancel{9}}}=\frac{11}{3}=3.\overline{6}$$

So, $\frac{33}{9}$ is rational and a repeating decimal.

c. $\sqrt{11}$: This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.

d. $\frac{17}{34}$: Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$\frac{17}{34}=\frac{\overset{1}{\cancel{17}}}{\underset{2}{\cancel{34}}}=\frac{1}{2}=0.5$$

So, $\frac{17}{34}$ is rational and a terminating decimal.

e. 0.3033033303333 … is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

Try It #3

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

a. $\frac{7}{77}$ **b.** $\sqrt{81}$ **c.** 4.27027002700027 … **d.** $\frac{91}{13}$ **e.** $\sqrt{39}$

Real Numbers

Given any number n, we know that n is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or −). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line. The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown in **Figure 2**.

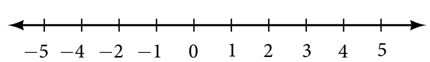


Figure 2 The real number line

Example 4 Classifying Real Numbers

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $-\frac{10}{3}$ **b.** $\sqrt{5}$ **c.** $-\sqrt{289}$ **d.** -6π **e.** 0.615384615384 …

Solution

a. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.

b. $\sqrt{5}$ is positive and irrational. It lies to the right of 0.

c. $-\sqrt{289}=-\sqrt{17^2}=-17$ is negative and rational. It lies to the left of 0.

d. -6π is negative and irrational. It lies to the left of 0.

e. 0.615384615384 … is a repeating decimal so it is rational and positive. It lies to the right of 0.

Try It #4

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

a. $\sqrt{73}$ **b.** $-11.411411411\ldots$ **c.** $\frac{47}{19}$ **d.** $-\frac{\sqrt{5}}{2}$ **e.** 6.210735

Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram, such as **Figure 3**.

N: the set of natural numbers
W: the set of whole numbers
I: the set of integers
Q: the set of rational numbers
Q': the set of irrational numbers

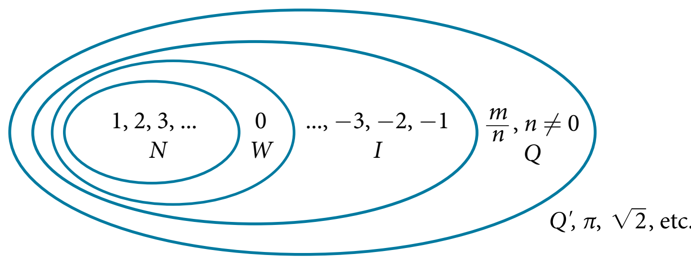


Figure 3 Sets of numbers

> ***sets of numbers***
>
> The set of **natural numbers** includes the numbers used for counting: $\{1, 2, 3, \ldots\}$.
>
> The set of **whole numbers** is the set of natural numbers plus zero: $\{0, 1, 2, 3, \ldots\}$.
>
> The set of **integers** adds the negative natural numbers to the set of whole numbers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
>
> The set of **rational numbers** includes fractions written as $\left\{\frac{m}{n} \middle| m \text{ and n are integers and } n \neq 0\right\}$
>
> The set of **irrational numbers** is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \mid h \text{ is not a rational number}\}$.

Example 5 **Differentiating the Sets of Numbers**

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $\sqrt{36}$ **b.** $\frac{8}{3}$ **c.** $\sqrt{73}$ **d.** -6 **e.** $3.2121121112\ldots$

Solution

	N	W	I	Q	Q'
a. $\sqrt{36} = 6$	×	×	×	×	
b. $\frac{8}{3} = 2.\overline{6}$				×	
c. $\sqrt{73}$					×
d. -6			×	×	
e. $3.2121121112\ldots$					×

Try It #5

Classify each number as being a natural number (N), whole number (W), integer (I), rational number (Q), and/or irrational number (Q').

a. $-\dfrac{35}{7}$ **b.** 0 **c.** $\sqrt{169}$ **d.** $\sqrt{24}$ **e.** $4.763763763\ldots$

Performing Calculations Using the Order of Operations

When we multiply a number by itself, we square it or raise it to a power of 2. For example, $4^2 = 4 \cdot 4 = 16$. We can raise any number to any power. In general, the **exponential notation** a^n means that the number or variable a is used as a factor n times.

$$a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$$

In this notation, a^n is read as the nth power of a, where a is called the **base** and n is called the **exponent**. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24 + 6 \cdot \dfrac{2}{3} - 4^2$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the **order of operations**. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces { } to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2, so simplify 4^2 as 16.

$$24 + 6 \cdot \frac{2}{3} - 4^2$$

$$24 + 6 \cdot \frac{2}{3} - 16$$

Next, perform multiplication or division, left to right.

$$24 + 6 \cdot \frac{2}{3} - 16$$

$$24 + 4 - 16$$

Lastly, perform addition or subtraction, left to right.

$$24 + 4 - 16$$

$$28 - 16$$

$$12$$

Therefore, $24 + 6 \cdot \dfrac{2}{3} - 4^2 = 12$.

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

order of operations

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym **PEMDAS**:

P(arentheses)

E(xponents)

M(ultiplication) and **D**(ivision)

A(ddition) and **S**(ubtraction)

How To...

Given a mathematical expression, simplify it using the order of operations.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Example 6 Using the Order of Operations

Use the order of operations to evaluate each of the following expressions.

a. $(3 \cdot 2)^2 - 4(6 + 2)$ **b.** $\dfrac{5^2 - 4}{7} - \sqrt{11 - 2}$ **c.** $6 - |5 - 8| + 3(4 - 1)$

d. $\dfrac{14 - 3 \cdot 2}{2 \cdot 5 - 3^2}$ **e.** $7(5 \cdot 3) - 2[(6 - 3) - 4^2] + 1$

Solution

a.
$$\begin{aligned} (3 \cdot 2)^2 - 4(6 + 2) &= (6)^2 - 4(8) && \text{Simplify parentheses.} \\ &= 36 - 4(8) && \text{Simplify exponent.} \\ &= 36 - 32 && \text{Simplify multiplication.} \\ &= 4 && \text{Simplify subtraction.} \end{aligned}$$

b.
$$\begin{aligned} \frac{5^2}{7} - \sqrt{11 - 2} &= \frac{5^2 - 4}{7} - \sqrt{9} && \text{Simplify grouping symbols (radical).} \\ &= \frac{5^2 - 4}{7} - 3 && \text{Simplify radical.} \\ &= \frac{25 - 4}{7} - 3 && \text{Simplify exponent.} \\ &= \frac{21}{7} - 3 && \text{Simplify subtraction in numerator.} \\ &= 3 - 3 && \text{Simplify division.} \\ &= 0 && \text{Simplify subtraction.} \end{aligned}$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.

c.
$$\begin{aligned} 6 - |5 - 8| + 3(4 - 1) &= 6 - |-3| + 3(3) && \text{Simplify inside grouping symbols.} \\ &= 6 - 3 + 3(3) && \text{Simplify absolute value.} \\ &= 6 - 3 + 9 && \text{Simplify multiplication.} \\ &= 3 + 9 && \text{Simplify subtraction.} \\ &= 12 && \text{Simplify addition.} \end{aligned}$$

d.
$$\begin{aligned} \frac{14 - 3 \cdot 2}{2 \cdot 5 - 3^2} &= \frac{14 - 3 \cdot 2}{2 \cdot 5 - 9} && \text{Simplify exponent.} \\ &= \frac{14 - 6}{10 - 9} && \text{Simplify products.} \\ &= \frac{8}{1} && \text{Simplify differences.} \\ &= 8 && \text{Simplify quotient.} \end{aligned}$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.

e
$$\begin{aligned} 7(5 \cdot 3) - 2[(6 - 3) - 4^2] + 1 &= 7(15) - 2[(3) - 4^2] + 1 && \text{Simplify inside parentheses.} \\ &= 7(15) - 2(3 - 16) + 1 && \text{Simplify exponent.} \\ &= 7(15) - 2(-13) + 1 && \text{Subtract.} \\ &= 105 + 26 + 1 && \text{Multiply.} \\ &= 132 && \text{Add.} \end{aligned}$$

Try It #6

Use the order of operations to evaluate each of the following expressions.

a. $\sqrt{5^2 - 4^2} + 7(5 - 4)^2$ **b.** $1 + \dfrac{7 \cdot 5 - 8 \cdot 4}{9 - 6}$ **c.** $|1.8 - 4.3| + 0.4\sqrt{15 + 10}$

d. $\frac{1}{2}[5 \cdot 3^2 - 7^2] + \frac{1}{3} \cdot 9^2$ **e.** $[(3 - 8)^2 - 4] - (3 - 8)$

Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.

$$a + b = b + a$$

We can better see this relationship when using real numbers.

$$(-2) + 7 = 5 \qquad \text{and} \qquad 7 + (-2) = 5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \cdot b = b \cdot a$$

Again, consider an example with real numbers.

$$(-11) \cdot (-4) = 44 \qquad \text{and} \qquad (-4) \cdot (-11) = 44$$

It is important to note that neither subtraction nor division is commutative. For example, $17 - 5$ is not the same as $5 - 17$. Similarly, $20 \div 5 \neq 5 \div 20$.

Associative Properties

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$a(bc) = (ab)c$$

Consider this example.

$$(3 \cdot 4) \cdot 5 = 60 \qquad \text{and} \qquad 3 \cdot (4 \cdot 5) = 60$$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c$$

This property can be especially helpful when dealing with negative integers. Consider this example.

$$[15 + (-9)] + 23 = 29 \qquad \text{and} \qquad 15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

$$\begin{aligned} 8 - (3 - 15) &\stackrel{?}{=} (8 - 3) - 15 \\ 8 - (-12) &\stackrel{?}{=} 5 - 15 \\ 20 &\neq -10 \end{aligned} \qquad\qquad \begin{aligned} 64 \div (8 \div 4) &\stackrel{?}{=} (64 \div 8) \div 4 \\ 64 \div 2 &\stackrel{?}{=} 8 \div 4 \\ 32 &\neq 2 \end{aligned}$$

As we can see, neither subtraction nor division is associative.

Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

$$\begin{aligned} 4 \cdot [12 + (-7)] &= \overbrace{4 \cdot 12} + \overbrace{4 \cdot (-7)} \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by -7, and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$\begin{aligned} 6 + (3 \cdot 5) &\stackrel{?}{=} (6 + 3) \cdot (6 + 5) \\ 6 + (15) &\stackrel{?}{=} (9) \cdot (11) \\ 21 &\neq 99 \end{aligned}$$

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b)$$

For example, consider the difference $12 - (5 + 3)$. We can rewrite the difference of the two terms 12 and $(5 + 3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5 + 3)$, we add the opposite.

$$12 + (-1) \cdot (5 + 3)$$

Now, distribute -1 and simplify the result.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-1) \cdot (5 + 3) \\ &= 12 + [(-1) \cdot 5 + (-1) \cdot 3] \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$\begin{aligned} 12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 + (-8) \\ &= 4 \end{aligned}$$

Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \cdot 1 = a$$

For example, we have $(-6) + 0 = -6$ and $23 \cdot 1 = 23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

Inverse Properties

The **inverse property of addition** states that, for every real number a, there is a unique number, called the additive inverse (or opposite), denoted$-a$, that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0$$

For example, if $a = -8$, the additive inverse is 8, since $(-8) + 8 = 0$.

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \cdot \frac{1}{a} = 1$$

For example, if $a = -\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$a \cdot \frac{1}{a} = \left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{2}\right) = 1$$

properties of real numbers

The following properties hold for real numbers a, b, and c.

	Addition	Multiplication
Commutative Property	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Property	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Distributive Property	$a \cdot (b + c) = a \cdot b + a \cdot c$	
Identity Property	There exists a unique real number called the additive identity, 0, such that, for any real number a $a + 0 = a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number a $a \cdot 1 = a$
Inverse Property	Every real number a has an additive inverse, or opposite, denoted $-a$, such that $a + (-a) = 0$	Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a \cdot \left(\frac{1}{a}\right) = 1$

Example 7 Using Properties of Real Numbers

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $3 \cdot 6 + 3 \cdot 4$ **b.** $(5 + 8) + (-8)$ **c.** $6 - (15 + 9)$ **d.** $\frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right)$ **e.** $100 \cdot [0.75 + (-2.38)]$

Solution

a.

$$\begin{aligned} 3 \cdot 6 + 3 \cdot 4 &= 3 \cdot (6 + 4) && \text{Distributive property} \\ &= 3 \cdot 10 && \text{Simplify.} \\ &= 30 && \text{Simplify.} \end{aligned}$$

b.

$$\begin{aligned} (5 + 8) + (-8) &= 5 + [8 + (-8)] && \text{Associative property of addition} \\ &= 5 + 0 && \text{Inverse property of addition} \\ &= 5 && \text{Identity property of addition} \end{aligned}$$

c.

$$\begin{aligned} 6 - (15 + 9) &= 6 + [(-15) + (-9)] && \text{Distributive property} \\ &= 6 + (-24) && \text{Simplify.} \\ &= -18 && \text{Simplify.} \end{aligned}$$

d.

$$\begin{aligned} \frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right) &= \frac{4}{7} \cdot \left(\frac{7}{4} \cdot \frac{2}{3}\right) && \text{Commutative property of multiplication} \\ &= \left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} && \text{Associative property of multiplication} \\ &= 1 \cdot \frac{2}{3} && \text{Inverse property of multiplication} \\ &= \frac{2}{3} && \text{Identity property of multiplication} \end{aligned}$$

e.

$$\begin{aligned} 100 \cdot [0.75 + (-2.38)] &= 100 \cdot 0.75 + 100 \cdot (-2.38) && \text{Distributive property} \\ &= 75 + (-238) && \text{Simplify.} \\ &= -163 && \text{Simplify.} \end{aligned}$$

Try It #7

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

a. $\left(-\frac{23}{5}\right) \cdot \left[11 \cdot \left(-\frac{5}{23}\right)\right]$ **b.** $5 \cdot (6.2 + 0.4)$ **c.** $18 - (7 - 15)$ **d.** $\frac{17}{18} + \left[\frac{4}{9} + \left(-\frac{17}{18}\right)\right]$ **e.** $6 \cdot (-3) + 6 \cdot 3$

Evaluating Algebraic Expressions

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x + 5$, $\frac{4}{3}\pi r^3$, or $\sqrt{2m^3n^2}$. In the expression $x + 5$, 5 is called a **constant** because it does not vary and x is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$(-3)^5 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) \qquad x^5 = x \cdot x \cdot x \cdot x \cdot x$$
$$(2 \cdot 7)^3 = (2 \cdot 7) \cdot (2 \cdot 7) \cdot (2 \cdot 7) \qquad (yz)^3 = (yz) \cdot (yz) \cdot (yz)$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

Example 8 Describing Algebraic Expressions

List the constants and variables for each algebraic expression.

a. $x + 5$ **b.** $\frac{4}{3}\pi r^3$ **c.** $\sqrt{2m^3n^2}$

Solution

	Constants	Variables
a. $x + 5$	5	x
b. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	r
c. $\sqrt{2m^3n^2}$	2	m, n

Try It #8

List the constants and variables for each algebraic expression.

a. $2\pi r(r + h)$ **b.** $2(L + W)$ **c.** $4y^3 + y$

Example 9 Evaluating an Algebraic Expression at Different Values

Evaluate the expression $2x - 7$ for each value for x.

a. $x = 0$ **b.** $x = 1$ **c.** $x = \frac{1}{2}$ **d.** $x = -4$

Solution

a. Substitute 0 for x.

$$\begin{aligned} 2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7 \end{aligned}$$

b. Substitute 1 for x.

$$\begin{aligned} 2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5 \end{aligned}$$

c. Substitute $\frac{1}{2}$ for x.

$$\begin{aligned} 2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6 \end{aligned}$$

d. Substitute -4 for x.

$$\begin{aligned} 2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15 \end{aligned}$$

Try It #9

Evaluate the expression $11 - 3y$ for each value for y.

a. $y = 2$ **b.** $y = 0$ **c.** $y = \frac{2}{3}$ **d.** $y = -5$

Example 10 Evaluating Algebraic Expressions

Evaluate each expression for the given values.

a. $x + 5$ for $x = -5$ **b.** $\frac{t}{2t-1}$ for $t = 10$ **c.** $\frac{4}{3}\pi r^3$ for $r = 5$

d. $a + ab + b$ for $a = 11, b = -8$ **e.** $\sqrt{2m^3n^2}$ for $m = 2, n = 3$

Solution

a. Substitute -5 for x.

$$\begin{aligned} x + 5 &= (-5) + 5 \\ &= 0 \end{aligned}$$

b. Substitute 10 for t.

$$\begin{aligned} \frac{t}{2t - 1} &= \frac{(10)}{2(10) - 1} \\ &= \frac{10}{20 - 1} \\ &= \frac{10}{19} \end{aligned}$$

c. Substitute 5 for r.

$$\begin{aligned} \frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi \end{aligned}$$

d. Substitute 11 for a and -8 for b.

$$\begin{aligned} a + ab + b &= (11) + (11)(-8) + (-8) \\ &= 11 - 88 - 8 \\ &= -85 \end{aligned}$$

e. Substitute 2 for m and 3 for n.

$$\begin{aligned} \sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Try It #10

Evaluate each expression for the given values.

a. $\frac{y + 3}{y - 3}$ for $y = 5$ **b.** $7 - 2t$ for $t = -2$ **c.** $\frac{1}{3}\pi r^2$ for $r = 11$

d. $(p^2q)^3$ for $p = -2, q = 3$ **e.** $4(m - n) - 5(n - m)$ for $m = \frac{2}{3}, n = \frac{1}{3}$

Formulas

An **equation** is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2x + 1 = 7$ has the unique solution $x = 3$ because when we substitute 3 for x in the equation, we obtain the true statement $2(3) + 1 = 7$.

A **formula** is an equation expressing a relationship between constant and variable quantities. Very often, the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other quantities. One of the most common examples is the formula for finding the area A of a circle in terms of the radius r of the circle: $A = \pi r^2$. For any value of r, the area A can be found by evaluating the expression πr^2.

Example 11 **Using a Formula**

A right circular cylinder with radius r and height h has the surface area S (in square units) given by the formula $S = 2\pi r(r + h)$. See **Figure 4**. Find the surface area of a cylinder with radius 6 in. and height 9 in. Leave the answer in terms of π.

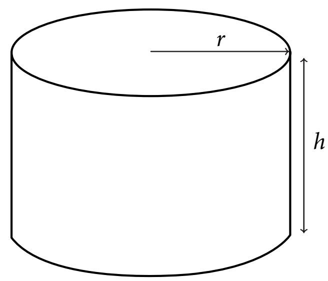


Figure 4 Right circular cylinder

Solution Evaluate the expression $2\pi r(r + h)$ for $r = 6$ and $h = 9$.

$$\begin{aligned} S &= 2\pi r(\mathrm{r} + \mathrm{h}) \\ &= 2\pi(6)[(6) + (9)] \\ &= 2\pi(6)(15) \\ &= 180\pi \end{aligned}$$

The surface area is 180π square inches.

Try It #11

A photograph with length L and width W is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or cm^2) is found to be $A = (L + 16)(W + 16) - L \cdot \mathrm{W}$. See **Figure 5**. Find the area of a matte for a photograph with length 32 cm and width 24 cm.

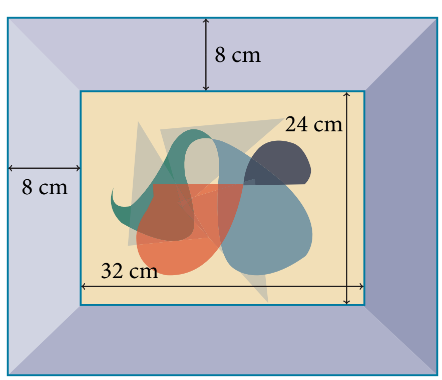


Figure 5

Simplifying Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

Example 12 **Simplifying Algebraic Expressions**

Simplify each algebraic expression.

a. $3x - 2y + x - 3y - 7$ **b.** $2r - 5(3 - r) + 4$ **c.** $\left(4t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right)$ **d.** $2mn - 5m + 3mn + n$

Solution

a.
$$\begin{aligned} 3x - 2y + x - 3y - 7 &= 3x + x - 2y - 3y - 7 && \text{Commutative property of addition} \\ &= 4x - 5y - 7 && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} 2r - 5(3 - r) + 4 &= 2r - 15 + 5r + 4 && \text{Distributive property} \\ &= 2r + 5y - 15 + 4 && \text{Commutative property of addition} \\ &= 7r - 11 && \text{Simplify.} \end{aligned}$$

c.
$$\begin{aligned} 4t - 4\left(t - \frac{5}{4}s\right) - \left(\frac{2}{3}t + 2s\right) &= 4t - \frac{5}{4}s - \frac{2}{3}t - 2s && \text{Distributive property} \\ &= 4t - \frac{2}{3}t - \frac{5}{4}s - 2s && \text{Commutative property of addition} \\ &= \frac{10}{3}t - \frac{13}{4}s && \text{Simplify.} \end{aligned}$$

d.
$$\begin{aligned} mn - 5m + 3mn + n &= 2mn + 3mn - 5m + n && \text{Commutative property of addition} \\ &= 5mn - 5m + n && \text{Simplify.} \end{aligned}$$

Try It #12

Simplify each algebraic expression.

a. $\frac{2}{3}y - 2\left(\frac{4}{3}y + z\right)$ **b.** $\frac{5}{t} - 2 - \frac{3}{t} + 1$ **c.** $4p(q - 1) + q(1 - p)$ **d.** $9r - (s + 2r) + (6 - s)$

Example 13 **Simplifying a Formula**

A rectangle with length L and width W has a perimeter P given by $P = L + W + L + W$. Simplify this expression.

Solution

$$\begin{aligned} P &= L + W + L + W && \\ P &= L + L + W + W && \text{Commutative property of addition} \\ P &= 2L + 2W && \text{Simplify.} \\ P &= 2(L + W) && \text{Distributive property} \end{aligned}$$

Try It #13

If the amount P is deposited into an account paying simple interest r for time t, the total value of the deposit A is given by $A = P + Prt$. Simplify the expression. (This formula will be explored in more detail later in the course.)

Access these online resources for additional instruction and practice with real numbers.

- Simplify an Expression (http://openstaxcollege.org/l/simexpress)
- Evaluate an Expression1 (http://openstaxcollege.org/l/ordofoper1)
- Evaluate an Expression2 (http://openstaxcollege.org/l/ordofoper2)

1.1 SECTION EXERCISES

VERBAL

1. Is $\sqrt{2}$ an example of a rational terminating, rational repeating, or irrational number? Tell why it fits that category.

2. What is the order of operations? What acronym is used to describe the order of operations, and what does it stand for?

3. What do the Associative Properties allow us to do when following the order of operations? Explain your answer.

NUMERIC

For the following exercises, simplify the given expression.

4. $10 + 2 \cdot (5 - 3)$

5. $6 \div 2 - (81 \div 3^2)$

6. $18 + (6 - 8)^3$

7. $-2 \cdot [16 \div (8 - 4)^2]^2$

8. $4 - 6 + 2 \cdot 7$

9. $3(5 - 8)$

10. $4 + 6 - 10 \div 2$

11. $12 \div (36 \div 9) + 6$

12. $(4 + 5)^2 \div 3$

13. $3 - 12 \cdot 2 + 19$

14. $2 + 8 \cdot 7 \div 4$

15. $5 + (6 + 4) - 11$

16. $9 - 18 \div 3^2$

17. $14 \cdot 3 \div 7 - 6$

18. $9 - (3 + 11) \cdot 2$

19. $6 + 2 \cdot 2 - 1$

20. $64 \div (8 + 4 \cdot 2)$

21. $9 + 4(2^2)$

22. $(12 \div 3 \cdot 3)^2$

23. $25 \div 5^2 - 7$

24. $(15 - 7) \cdot (3 - 7)$

25. $2 \cdot 4 - 9(-1)$

26. $4^2 - 25 \cdot \frac{1}{5}$

27. $12(3 - 1) \div 6$

ALGEBRAIC

For the following exercises, solve for the variable.

28. $8(x + 3) = 64$

29. $4y + 8 = 2y$

30. $(11a + 3) - 18a = -4$

31. $4z - 2z(1 + 4) = 36$

32. $4y(7 - 2)^2 = -200$

33. $-(2x)^2 + 1 = -3$

34. $8(2 + 4) - 15b = b$

35. $2(11c - 4) = 36$

36. $4(3 - 1)x = 4$

37. $\frac{1}{4}(8w - 4^2) = 0$

For the following exercises, simplify the expression.

38. $4x + x(13 - 7)$

39. $2y - (4)^2 y - 11$

40. $\frac{a}{2^3}(64) - 12a \div 6$

41. $8b - 4b(3) + 1$

42. $5l \div 3l \cdot (9 - 6)$

43. $7z - 3 + z \cdot 6^2$

44. $4 \cdot 3 + 18x \div 9 - 12$

45. $9(y + 8) - 27$

46. $\left(\frac{9}{6}t - 4\right)2$

47. $6 + 12b - 3 \cdot 6b$

48. $18y - 2(1 + 7y)$

49. $\left(\frac{4}{9}\right)^2 \cdot 27x$

50. $8(3 - m) + 1(-8)$

51. $9x + 4x(2 + 3) - 4(2x + 3x)$

52. $5^2 - 4(3x)$

REAL-WORLD APPLICATIONS

For the following exercises, consider this scenario: Fred earns $40 mowing lawns. He spends $10 on mp3s, puts half of what is left in a savings account, and gets another $5 for washing his neighbor's car.

53. Write the expression that represents the number of dollars Fred keeps (and does not put in his savings account). Remember the order of operations.

54. How much money does Fred keep?

For the following exercises, solve the given problem.

55. According to the U.S. Mint, the diameter of a quarter is 0.955 inches. The circumference of the quarter would be the diameter multiplied by π. Is the circumference of a quarter a whole number, a rational number, or an irrational number?

56. Jessica and her roommate, Adriana, have decided to share a change jar for joint expenses. Jessica put her loose change in the jar first, and then Adriana put her change in the jar. We know that it does not matter in which order the change was added to the jar. What property of addition describes this fact?

For the following exercises, consider this scenario: There is a mound of g pounds of gravel in a quarry. Throughout the day, 400 pounds of gravel is added to the mound. Two orders of 600 pounds are sold and the gravel is removed from the mound. At the end of the day, the mound has 1,200 pounds of gravel.

57. Write the equation that describes the situation.

58. Solve for g.

For the following exercise, solve the given problem.

59. Ramon runs the marketing department at his company. His department gets a budget every year, and every year, he must spend the entire budget without going over. If he spends less than the budget, then his department gets a smaller budget the following year. At the beginning of this year, Ramon got $2.5 million for the annual marketing budget. He must spend the budget such that $2{,}500{,}000 - x = 0$. What property of addition tells us what the value of x must be?

TECHNOLOGY

For the following exercises, use a graphing calculator to solve for x. Round the answers to the nearest hundredth.

60. $0.5(12.3)^2 - 48x = \frac{3}{5}$

61. $(0.25 - 0.75)^2x - 7.2 = 9.9$

EXTENSIONS

62. If a whole number is not a natural number, what must the number be?

63. Determine whether the statement is true or false: The multiplicative inverse of a rational number is also rational.

64. Determine whether the statement is true or false: The product of a rational and irrational number is always irrational.

65. Determine whether the simplified expression is rational or irrational: $\sqrt{-18 - 4(5)(-1)}$.

66. Determine whether the simplified expression is rational or irrational: $\sqrt{-16 + 4(5) + 5}$.

67. The division of two whole numbers will always result in what type of number?

68. What property of real numbers would simplify the following expression: $4 + 7(x - 1)$?

LEARNING OBJECTIVES

In this section students will:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

1.2 EXPONENTS AND SCIENTIFIC NOTATION

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2{,}048 \cdot 1{,}536 \cdot 48 \cdot 24 \cdot 3{,}600$ and press **ENTER**. The calculator displays **1.304596316E13**. What does this mean? The "**E13**" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately $1.3 \cdot 10^{13}$ bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

Using the Product Rule of Exponents

Consider the product $x^3 \cdot x^4$. Both terms have the same base, x, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$\begin{aligned} x^3 \cdot x^4 &= \overbrace{x \cdot x \cdot x}^{\text{3 factors}} \cdot \overbrace{x \cdot x \cdot x \cdot x}^{\text{4 factors}} \\ &= \overbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}^{\text{7 factors}} \\ &= x^7 \end{aligned}$$

The result is that $x^3 \cdot x^4 = x^{3+4} = x^7$.

Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the *product rule of exponents*.

$$a^m \cdot a^n = a^{m+n}$$

Now consider an example with real numbers.

$$2^3 \cdot 2^4 = 2^{3+4} = 2^7$$

We can always check that this is true by simplifying each exponential expression. We find that 2^3 is 8, 2^4 is 16, and 2^7 is 128. The product $8 \cdot 16$ equals 128, so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

the product rule of exponents

For any real number a and natural numbers m and n, the product rule of exponents states that

$$a^m \cdot a^n = a^{m+n}$$

Example 1 Using the Product Rule

Write each of the following products with a single base. Do not simplify further.

a. $t^5 \cdot t^3$ **b.** $(-3)^5 \cdot (-3)$ **c.** $x^2 \cdot x^5 \cdot x^3$

Solution Use the product rule to simplify each expression.

a. $t^5 \cdot t^3 = t^{5+3} = t^8$

b. $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$

c. $x^2 \cdot x^5 \cdot x^3$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

Try It #1

Write each of the following products with a single base. Do not simplify further.

a. $k^6 \cdot k^9$ **b.** $\left(\frac{2}{y}\right)^4 \cdot \left(\frac{2}{y}\right)$ **c.** $t^3 \cdot t^6 \cdot t^5$

Using the Quotient Rule of Exponents

The *quotient rule of exponents* allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^m}{y^n}$, where $m > n$. Consider the example $\frac{y^9}{y^5}$. Perform the division by canceling common factors.

$$\begin{aligned}\frac{y^9}{y^5} &= \frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\ &= \frac{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot y \cdot y \cdot y \cdot y}{\cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}} \\ &= \frac{y \cdot y \cdot y \cdot y}{1} \\ &= y^4\end{aligned}$$

Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$\frac{a^m}{a^n} = a^{m-n}$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$\frac{y^9}{y^5} = y^{9-5} = y^4$$

For the time being, we must be aware of the condition $m > n$. Otherwise, the difference $m - n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

the quotient rule of exponents

For any real number a and natural numbers m and n, such that $m > n$, the quotient rule of exponents states that

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 2 Using the Quotient Rule

Write each of the following products with a single base. Do not simplify further.

a. $\frac{(-2)^{14}}{(-2)^9}$ **b.** $\frac{t^{23}}{t^{15}}$ **c.** $\frac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}}$

Solution Use the quotient rule to simplify each expression.

a. $\dfrac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$

b. $\dfrac{t^{23}}{t^{15}} = t^{23-15} = t^8$

c. $\dfrac{\left(z\sqrt{2}\right)^5}{z\sqrt{2}} = \left(z\sqrt{2}\right)^{5-1} = \left(z\sqrt{2}\right)^4$

Try It #2

Write each of the following products with a single base. Do not simplify further.

a. $\dfrac{s^{75}}{s^{68}}$ **b.** $\dfrac{(-3)^6}{-3}$ **c.** $\dfrac{(ef^2)^5}{(ef^2)^3}$

Using the Power Rule of Exponents

Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $(x^2)^3$. The expression inside the parentheses is multiplied twice because it has an exponent of 2. Then the result is multiplied three times because the entire expression has an exponent of 3.

$$\begin{aligned}(x^2)^3 &= \overbrace{(x^2)\cdot(x^2)\cdot(x^2)}^{\text{3 factors}}\\ &= \overbrace{\Big(\overbrace{x\cdot x}^{\text{2 factors}}\Big)\cdot\Big(\overbrace{x\cdot x}^{\text{2 factors}}\Big)\cdot\Big(\overbrace{x\cdot x}^{\text{2 factors}}\Big)}^{\text{3 factors}}\\ &= x\cdot x\cdot x\cdot x\cdot x\cdot x\\ &= x^6\end{aligned}$$

The exponent of the answer is the product of the exponents: $(x^2)^3 = x^{2\cdot 3} = x^6$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$(a^m)^n = a^{m\cdot n}$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

	Product Rule						**Power Rule**			
$5^3\cdot 5^4$	$=$	5^{3+4}	$=$	5^7	but	$(5^3)^4$	$=$	$5^{3\cdot 4}$	$=$	5^{12}
$x^5\cdot x^2$	$=$	x^{5+2}	$=$	x^7	but	$(x^5)^2$	$=$	$x^{5\cdot 2}$	$=$	x^{10}
$(3a)^7\cdot(3a)^{10}$	$=$	$(3a)^{7+10}$	$=$	$(3a)^{17}$	but	$((3a)^7)^{10}$	$=$	$(3a)^{7\cdot 10}$	$=$	$(3a)^{70}$

the power rule of exponents

For any real number a and positive integers m and n, the power rule of exponents states that

$$(a^m)^n = a^{m\cdot n}$$

Example 3 **Using the Power Rule**

Write each of the following products with a single base. Do not simplify further.

a. $(x^2)^7$ **b.** $((2t)^5)^3$ **c.** $((-3)^5)^{11}$

Solution Use the power rule to simplify each expression.

a. $(x^2)^7 = x^{2\cdot 7} = x^{14}$

b. $((2t)^5)^3 = (2t)^{5\cdot 3} = (2t)^{15}$

c. $((-3)^5)^{11} = (-3)^{5\cdot 11} = (-3)^{55}$

Try It #3

Write each of the following products with a single base. Do not simplify further.

a. $((3y)^8)^3$ **b.** $(t^5)^7$ **c.** $((-g)^4)^4$

Using the Zero Exponent Rule of Exponents

Return to the quotient rule. We made the condition that $m > n$ so that the difference $m - n$ would never be zero or negative. What would happen if $m = n$? In this case, we would use the zero exponent rule of exponents to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is $t^0 = 1$. This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression 0^0. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

the zero exponent rule of exponents

For any nonzero real number a, the zero exponent rule of exponents states that

$$a^0 = 1$$

Example 4 Using the Zero Exponent Rule

Simplify each expression using the zero exponent rule of exponents.

a. $\dfrac{c^3}{c^3}$ **b.** $\dfrac{-3x^5}{x^5}$ **c.** $\dfrac{(j^2k)^4}{(j^2k)\cdot(j^2k)^3}$ **d.** $\dfrac{5(rs^2)^2}{(rs^2)^2}$

Solution Use the zero exponent and other rules to simplify each expression.

a.
$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3}\\ &= c^0\end{aligned}$$

b.
$$\begin{aligned}\frac{-3x^5}{x^5} &= -3\cdot\frac{x^5}{x^5}\\ &= -3\cdot x^{5-5}\\ &= -3\cdot x^0\\ &= -3\cdot 1\\ &= -3\end{aligned}$$

c.
$$\begin{aligned}\frac{(j^2k)^4}{(j^2k)\cdot(j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator.}\\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify.}\\ &= (j^2k)^{4-4} && \text{Use the quotient rule.}\\ &= (j^2k)^0 && \text{Simplify.}\\ &= 1\end{aligned}$$

d.
$$\begin{aligned}\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule.}\\ &= 5(rs^2)^0 && \text{Simplify.}\\ &= 5\cdot 1 && \text{Use the zero exponent rule.}\\ &= 5 && \text{Simplify.}\end{aligned}$$

Try It #4

Simplify each expression using the zero exponent rule of exponents.

a. $\dfrac{t^7}{t^7}$ **b.** $\dfrac{(de^2)^{11}}{2(de^2)^{11}}$ **c.** $\dfrac{w^4 \cdot w^2}{w^6}$ **d.** $\dfrac{t^3 \cdot t^4}{t^2 \cdot t^5}$

Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m > n$ in the quotient rule even further. For example, can we simplify $\dfrac{h^3}{h^5}$? When $m < n$ —that is, where the difference $m - n$ is negative—we can use the negative rule of exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\dfrac{h^3}{h^5}$.

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h}\\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h}\\ &= \frac{1}{h \cdot h}\\ &= \frac{1}{h^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5}\\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have $h^{-2} = \dfrac{1}{h^2}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \text{ and } a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression a^n is defined when n is a natural number, 0, or the negative of a natural number. That means that an is defined for any integer n. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer n.

> ***the negative rule of exponents***
>
> For any nonzero real number a and natural number n, the negative rule of exponents states that
>
> $$a^{-n} = \frac{1}{a^n}$$

Example 5 **Using the Negative Exponent Rule**

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\dfrac{\theta^3}{\theta^{10}}$ **b.** $\dfrac{z^2 \cdot z}{z^4}$ **c.** $\dfrac{(-5t^3)^4}{(-5t^3)^8}$

Solution

a. $\dfrac{\theta^3}{\theta^{10}} = \theta^{3-10} = \theta^{-7} = \dfrac{1}{\theta^7}$

b. $\dfrac{z^2 \cdot z}{z^4} = \dfrac{z^{2+1}}{z^4} = \dfrac{z^3}{z^4} = z^{3-4} = z^{-1} = \dfrac{1}{z}$

c. $\dfrac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \dfrac{1}{(-5t^3)^4}$

Try It #5

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

a. $\dfrac{(-3t)^2}{(-3t)^8}$ **b.** $\dfrac{f^{47}}{f^{49} \cdot f}$ **c.** $\dfrac{2k^4}{5k^7}$

Example 6 Using the Product and Quotient Rules

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $b^2 \cdot b^{-8}$ **b.** $(-x)^5 \cdot (-x)^{-5}$ **c.** $\dfrac{-7z}{(-7z)^5}$

Solution

a. $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \dfrac{1}{b^6}$

b. $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$

c. $\dfrac{-7z}{(-7z)^5} = \dfrac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \dfrac{1}{(-7z)^4}$

Try It #6

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

a. $t^{-11} \cdot t^6$ **b.** $\dfrac{25^{12}}{25^{13}}$

Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents*, which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider $(pq)^3$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= \overbrace{(pq) \cdot (pq) \cdot (pq)}^{3 \text{ factors}} \\ &= p \cdot q \cdot p \cdot q \cdot p \cdot q \\ &= \overbrace{p \cdot p \cdot p}^{3 \text{ factors}} \cdot \overbrace{q \cdot q \cdot q}^{3 \text{ factors}} \\ &= p^3 \cdot q^3\end{aligned}$$

In other words, $(pq)^3 = p^3 \cdot q^3$.

> ***the power of a product rule of exponents***
> For any nonzero real number a and natural number n, the negative rule of exponents states that
> $$(ab)^n = a^n b^n$$

Example 7 Using the Power of a Product Rule

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(ab^2)^3$ **b.** $(2t)^{15}$ **c.** $(-2w^3)^3$ **d.** $\dfrac{1}{(-7z)^4}$ **e.** $(e^{-2}f^2)^7$

Solution Use the product and quotient rules and the new definitions to simplify each expression.

a. $(ab^2)^3 = (a)^3 \cdot (b^2)^3 = a^{1 \cdot 3} \cdot b^{2 \cdot 3} = a^3b^6$

b. $(2t)^{15} = (2)^{15} \cdot (t)^{15} = 2^{15}t^{15} = 32{,}768t^{15}$

c. $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3 \cdot 3} = -8w^9$

d. $\dfrac{1}{(-7z)^4} = \dfrac{1}{(-7)^4 \cdot (z)^4} = \dfrac{1}{2{,}401z^4}$

e. $(e^{-2}f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2 \cdot 7} \cdot f^{2 \cdot 7} = e^{-14}f^{14} = \dfrac{f^{14}}{e^{14}}$

Try It #7

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

a. $(g^2h^3)^5$ **b.** $(5t)^3$ **c.** $(-3y^5)^3$ **d.** $\dfrac{1}{(a^6b^7)^3}$ **e.** $(r^3s^{-2})^4$

Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$(e^{-2}f^2)^7 = \frac{f^{14}}{e^{14}}$$

Let's rewrite the original problem differently and look at the result.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}(e^{-2}f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7\\ &= \frac{(f^2)^7}{(e^2)^7}\\ &= \frac{f^{2\cdot 7}}{e^{2\cdot 7}}\\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

the power of a quotient rule of exponents

For any real numbers a and b and any integer n, the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example 8 **Using the Power of a Quotient Rule**

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\dfrac{4}{z^{11}}\right)^3$ **b.** $\left(\dfrac{p}{q^3}\right)^6$ **c.** $\left(\dfrac{-1}{t^2}\right)^{27}$ **d.** $(j^3k^{-2})^4$ **e.** $(m^{-2}n^{-2})^3$

Solution

a. $\left(\dfrac{4}{z^{11}}\right)^3 = \dfrac{4^3}{(z^{11})^3} = \dfrac{64}{z^{11\cdot 3}} = \dfrac{64}{z^{33}}$

b. $\left(\dfrac{p}{q^3}\right)^6 = \dfrac{p^6}{(q^3)^6} = \dfrac{p^{1\cdot 6}}{q^{3\cdot 6}} = \dfrac{p^6}{q^{18}}$

c. $\left(\dfrac{-1}{t^2}\right)^{27} = \dfrac{(-1)^{27}}{(t^2)^{27}} = \dfrac{-1}{t^{2\cdot 27}} = \dfrac{-1}{t^{54}} = \dfrac{-1}{t^{54}}$

d. $(j^3k^{-2})^4 = \left(\dfrac{j^3}{k^2}\right)^4 = \dfrac{(j^3)^4}{(k^2)^4} = \dfrac{j^{3\cdot 4}}{k^{2\cdot 4}} = \dfrac{j^{12}}{k^8}$

e. $(m^{-2}n^{-2})^3 = \left(\dfrac{1}{m^2n^2}\right)^3 = \left(\dfrac{1^3}{(m^2n^2)^3}\right) = \dfrac{1}{(m^2)^3(n^2)^3} = \dfrac{1}{m^{2\cdot 3}\cdot n^{2\cdot 3}} = \dfrac{1}{m^6n^6}$

Try It #8

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

a. $\left(\frac{b^5}{c}\right)^3$ **b.** $\left(\frac{5}{u^8}\right)^4$ **c.** $\left(\frac{-1}{w^3}\right)^{35}$ **d.** $(p^{-4}q^3)^8$ **e.** $(c^{-5}d^{-3})^4$

Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

Example 9 Simplifying Exponential Expressions

Simplify each expression and write the answer with positive exponents only.

a. $(6m^2n^{-1})^3$ **b.** $17^5 \cdot 17^{-4} \cdot 17^{-3}$ **c.** $\left(\frac{u^{-1}v}{v^{-1}}\right)^2$ **d.** $(-2a^3b^{-1})(5a^{-2}b^2)$

e. $\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4}$ **f.** $\frac{(3w^2)^5}{(6w^{-2})^2}$

Solution

a. $(6m^2n^{-1})^3 = (6)^3(m^2)^3(n^{-1})^3$ The power of a product rule

$= 6^3m^{2\cdot3}n^{-1\cdot3}$ The power rule

$= 216m^6n^{-3}$ Simplify.

$= \frac{216m^6}{n^3}$ The negative exponent rule

b. $17^5 \cdot 17^{-4} \cdot 17^{-3} = 17^{5-4-3}$ The product rule

$= 17^{-2}$ Simplify.

$= \frac{1}{17^2}$ or $\frac{1}{289}$ The negative exponent rule

c. $\left(\frac{u^{-1}v}{v^{-1}}\right)^2 = \frac{(u^{-1}v)^2}{(v^{-1})^2}$ The power of a quotient rule

$= \frac{u^{-2}v^2}{v^{-2}}$ The power of a product rule

$= u^{-2}v^{2-(-2)}$ The quotient rule

$= u^{-2}v^4$ Simplify.

$= \frac{v^4}{u^2}$ The negative exponent rule

d. $(-2a^3b^{-1})(5a^{-2}b^2) = -2\cdot5\cdot a^3\cdot a^{-2}\cdot b^{-1}\cdot b^2$ Commutative and associative laws of multiplication

$= -10\cdot a^{3-2}\cdot b^{-1+2}$ The product rule

$= -10ab$ Simplify.

e. $\left(x^2\sqrt{2}\right)^4\left(x^2\sqrt{2}\right)^{-4} = \left(x^2\sqrt{2}\right)^{4-4}$ The product rule

$= \left(x^2\sqrt{2}\right)^0$ Simplify.

$= 1$ The zero exponent rule

f. $\frac{(3w^2)^5}{(6w^{-2})^2} = \frac{3^5\cdot(w^2)^5}{6^2\cdot(w^{-2})^2}$ The power of a product rule

$= \frac{3^5w^{2\cdot5}}{6^2w^{-2\cdot2}}$ The power rule

$= \frac{243w^{10}}{36w^{-4}}$ Simplify.

$= \frac{27w^{10-(-4)}}{4}$ The quotient rule and reduce fraction

$= \frac{27w^{14}}{4}$ Simplify.

Try It #9

Simplify each expression and write the answer with positive exponents only.

a. $(2uv^{-2})^{-3}$ **b.** $x^8 \cdot x^{-12} \cdot x$ **c.** $\left(\dfrac{e^2f^{-3}}{f^{-1}}\right)^2$ **d.** $(9r^{-5}s^3)(3r^6s^{-4})$ **e.** $\left(\dfrac{4}{9}tw^{-2}\right)^{-3}\left(\dfrac{4}{9}tw^{-2}\right)^3$ **f.** $\dfrac{(2h^2k)^4}{(7h^{-1}k^2)^2}$

Using Scientific Notation

Recall at the beginning of the section that we found the number 1.3×10^{13} when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m, and the radius of an electron, which is about 0.00000000000047 m. How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called **scientific notation**, in which we express numbers in terms of exponents of 10. To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places n that you moved the decimal point. Multiply the decimal number by 10 raised to a power of n. If you moved the decimal left as in a very large number, n is positive. If you moved the decimal right as in a small large number, n is negative.

For example, consider the number 2,780,418. Move the decimal left until it is to the right of the first nonzero digit, which is 2.

$$2{,}780418 \longrightarrow 2.\overbrace{780418}^{\text{6 places left}}$$

We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6, and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$2.780418 \times 10^6$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m. Perform the same series of steps as above, except move the decimal point to the right.

$$0.00000000000047 \longrightarrow 0\overbrace{0000000000004}^{\text{13 places right}}.7$$

Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13. The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$4.7 \times 10^{-13}$$

> ***scientific notation***
> A number is written in scientific notation if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

Example 10 **Converting Standard Notation to Scientific Notation**

Write each number in scientific notation.

a. Distance to Andromeda Galaxy from Earth: 24,000,000,000,000,000,000,000 m
b. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
c. Number of stars in Andromeda Galaxy: 1,000,000,000,000
d. Diameter of electron: 0.00000000000094 m
e. Probability of being struck by lightning in any single year: 0.00000143

Solution

a. 24,000,000,000,000,000,000,000 m
$\longleftarrow$ 22 places
2.4×10^{22} m

b. 1,300,000,000,000,000,000,000 m
$\longleftarrow$ 21 places
1.3×10^{21} m

c. 1,000,000,000,000
$\longleftarrow$ 12 places
1×10^{12}

d. 0.00000000000094 m
$\longrightarrow$ 13 places
9.4×10^{-13} m

e. 0.00000143
$\longrightarrow$ 6 places
1.43×10^{-6}

Analysis *Observe that, if the given number is greater than* 1, *as in examples a–c, the exponent of* 10 *is positive; and if the number is less than* 1, *as in examples d–e, the exponent is negative.*

Try It #10

Write each number in scientific notation.

a. U.S. national debt per taxpayer (April 2014): $152,000
b. World population (April 2014): 7,158,000,000
c. World gross national income (April 2014): $85,500,000,000,000
d. Time for light to travel 1 m: 0.00000000334 s
e. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Converting from Scientific to Standard Notation

To convert a number in **scientific notation** to standard notation, simply reverse the process. Move the decimal n places to the right if n is positive or n places to the left if n is negative and add zeros as needed. Remember, if n is positive, the value of the number is greater than 1, and if n is negative, the value of the number is less than one.

Example 11 Converting Scientific Notation to Standard Notation

Convert each number in scientific notation to standard notation.

a. 3.547×10^{14} **b.** -2×10^{6} **c.** 7.91×10^{-7} **d.** -8.05×10^{-12}

Solution

a. 3.547×10^{14}
3.54700000000000
$\longrightarrow$ 14 places
354,700,000,000,000

b. -2×10^{6}
-2.000000
$\longrightarrow$ 6 places
–2,000,000

c. 7.91×10^{-7}
0000007.91
$\longleftarrow$ 7 places
0.000000791

d. -8.05×10^{-12}
-000000000008.05
$\longleftarrow$ 12 places
–0.00000000000805

Try It #11

Convert each number in scientific notation to standard notation.

a. 7.03×10^5 **b.** -8.16×10^{11} **c.** -3.9×10^{-13} **d.** 8×10^{-6}

Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms (2 hydrogen and 1 oxygen). The average drop of water contains around 1.32×10^{21} molecules of water and 1 L of water holds about 1.22×10^4 average drops. Therefore, there are approximately $3 \times (1.32 \times 10^{21}) \times (1.22 \times 10^4) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $(7 \times 10^4) \times (5 \times 10^6) = 35 \times 10^{10}$. The answer is not in proper scientific notation because 35 is greater than 10. Consider 35 as 3.5×10. That adds a ten to the exponent of the answer.

$$(35) \times 10^{10} = (3.5 \times 10) \times 10^{10} = 3.5 \times (10 \times 10^{10}) = 3.5 \times 10^{11}$$

Example 12 **Using Scientific Notation**

Perform the operations and write the answer in scientific notation.

a. $(8.14 \times 10^{-7})\,(6.5 \times 10^{10})$

b. $(4 \times 10^5) \div (-1.52 \times 10^9)$

c. $(2.7 \times 10^5)\,(6.04 \times 10^{13})$

d. $(1.2 \times 10^8) \div (9.6 \times 10^5)$

e. $(3.33 \times 10^4)\,(-1.05 \times 10^7)\,(5.62 \times 10^5)$

Solution

a.
$$\begin{aligned}(8.14 \times 10^{-7})\,(6.5 \times 10^{10}) &= (8.14 \times 6.5)\,(10^{-7} \times 10^{10}) && \text{Commutative and associative properties of multiplication}\\ &= (52.91)\,(10^3) && \text{Product rule of exponents}\\ &= 5.291 \times 10^4 && \text{Scientific notation}\end{aligned}$$

b.
$$\begin{aligned}(4 \times 10^5) \div (-1.52 \times 10^9) &= \left(\frac{4}{-1.52}\right)\left(\frac{10^5}{10^9}\right) && \text{Commutative and associative properties of multiplication}\\ &\approx (-2.63)\,(10^{-4}) && \text{Quotient rule of exponents}\\ &= -2.63 \times 10^{-4} && \text{Scientific notation}\end{aligned}$$

c.
$$\begin{aligned}(2.7 \times 10^5)\,(6.04 \times 10^{13}) &= (2.7 \times 6.04)\,(10^5 \times 10^{13}) && \text{Commutative and associative properties of multiplication}\\ &= (16.308)\,(10^{18}) && \text{Product rule of exponents}\\ &= 1.6308 \times 10^{19} && \text{Scientific notation}\end{aligned}$$

d.
$$\begin{aligned}(1.2 \times 10^8) \div (9.6 \times 10^5) &= \left(\frac{1.2}{9.6}\right)\left(\frac{10^8}{10^5}\right) && \text{Commutative and associative properties of multiplication}\\ &= (0.125)\,(10^3) && \text{Quotient rule of exponents}\\ &= 1.25 \times 10^2 && \text{Scientific notation}\end{aligned}$$

e.
$$\begin{aligned}(3.33 \times 10^4)(-1.05 \times 10^7)\,(5.62 \times 10^5) &= [3.33 \times (-1.05) \times 5.62]\,(10^4 \times 10^7 \times 10^5)\\ &\approx (-19.65)\,(10^{16})\\ &= -1.965 \times 10^{17}\end{aligned}$$

Try It #12

Perform the operations and write the answer in scientific notation.

a. $(-7.5 \times 10^8)(1.13 \times 10^{-2})$

b. $(1.24 \times 10^{11}) \div (1.55 \times 10^{18})$

c. $(3.72 \times 10^9)(8 \times 10^3)$

d. $(9.933 \times 10^{23}) \div (-2.31 \times 10^{17})$

e. $(-6.04 \times 10^9)(7.3 \times 10^2)(-2.81 \times 10^2)$

Example 13 Applying Scientific Notation to Solve Problems

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about \$17,547,000,000,000. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

Solution The population was $308{,}000{,}000 = 3.08 \times 10^8$.

The national debt was $\$17{,}547{,}000{,}000{,}000 \approx \$1.75 \times 10^{13}$.

To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$\begin{aligned}(1.75 \times 10^{13}) \div (3.08 \times 10^8) &= \left(\frac{1.75}{3.08}\right) \times \left(\frac{10^{13}}{10^8}\right)\\ &\approx 0.57 \times 10^5\\ &= 5.7 \times 10^4\end{aligned}$$

The debt per citizen at the time was about $\$5.7 \times 10^4$, or \$57,000.

Try It #13

An average human body contains around 30,000,000,000,000 red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.

Access these online resources for additional instruction and practice with exponents and scientific notation.

- Exponential Notation (http://openstaxcollege.org/l/exponnot)
- Properties of Exponents (http://openstaxcollege.org/l/exponprops)
- Zero Exponent (http://openstaxcollege.org/l/zeroexponent)
- Simplify Exponent Expressions (http://openstaxcollege.org/l/exponexpres)
- Quotient Rule for Exponents (http://openstaxcollege.org/l/quotofexpon)
- Scientific Notation (http://openstaxcollege.org/l/scientificnota)
- Converting to Decimal Notation (http://openstaxcollege.org/l/decimalnota)

1.2 SECTION EXERCISES

VERBAL

1. Is 2^3 the same as 3^2? Explain.

2. When can you add two exponents?

3. What is the purpose of scientific notation?

4. Explain what a negative exponent does.

NUMERIC

For the following exercises, simplify the given expression. Write answers with positive exponents.

5. 9^2

6. 15^{-2}

7. $3^2 \cdot 3^3$

8. $4^4 \div 4$

9. $(2^2)^{-2}$

10. $(5 - 8)^0$

11. $11^3 \div 11^4$

12. $6^5 \cdot 6^{-7}$

13. $(8^0)^2$

14. $5^{-2} \div 5^2$

For the following exercises, write each expression with a single base. Do not simplify further. Write answers with positive exponents.

15. $4^2 \cdot 4^3 \div 4^{-4}$

16. $\dfrac{6^{12}}{6^9}$

17. $(12^3 \cdot 12)^{10}$

18. $10^6 \div (10^{10})^{-2}$

19. $7^{-6} \cdot 7^{-3}$

20. $(3^3 \div 3^4)^5$

For the following exercises, express the decimal in scientific notation.

21. 0.0000314

22. 148,000,000

For the following exercises, convert each number in scientific notation to standard notation.

23. 1.6×10^{10}

24. 9.8×10^{-9}

ALGEBRAIC

For the following exercises, simplify the given expression. Write answers with positive exponents.

25. $\dfrac{a^3a^2}{a}$

26. $\dfrac{mn^2}{m^{-2}}$

27. $(b^3c^4)^2$

28. $\left(\dfrac{x^{-3}}{y^2}\right)^{-5}$

29. $ab^2 \div d^{-3}$

30. $(w^0x^5)^{-1}$

31. $\dfrac{m^4}{n^0}$

32. $y^{-4}(y^2)^2$

33. $\dfrac{p^{-4}q^2}{p^2q^{-3}}$

34. $(l \times w)^2$

35. $(y^7)^3 \div x^{14}$

36. $\left(\dfrac{a}{2^3}\right)^2$

37. $5^2m \div 5^0m$

38. $\dfrac{(16\sqrt{x})^2}{y^{-1}}$

39. $\dfrac{2^3}{(3a)^{-2}}$

40. $(ma^6)^2\dfrac{1}{m^3a^2}$

41. $(b^{-3}c)^3$

42. $(x^2y^{13} \div y^0)^2$

43. $(9z^3)^{-2}y$

REAL-WORLD APPLICATIONS

44. To reach escape velocity, a rocket must travel at the rate of 2.2×10^6 ft/min. Rewrite the rate in standard notation.

45. A dime is the thinnest coin in U.S. currency. A dime's thickness measures 2.2×10^6 m. Rewrite the number in standard notation.

46. The average distance between Earth and the Sun is 92,960,000 mi. Rewrite the distance using scientific notation.

47. A terabyte is made of approximately 1,099,500,000,000 bytes. Rewrite in scientific notation.

48. The Gross Domestic Product (GDP) for the United States in the first quarter of 2014 was $\$1.71496 \times 10^{13}$. Rewrite the GDP in standard notation.

49. One picometer is approximately 3.397×10^{-11} in. Rewrite this length using standard notation.

50. The value of the services sector of the U.S. economy in the first quarter of 2012 was \$10,633.6 billion. Rewrite this amount in scientific notation.

TECHNOLOGY

For the following exercises, use a graphing calculator to simplify. Round the answers to the nearest hundredth.

51. $\left(\dfrac{12^3\,m^{33}}{4^{-3}}\right)^2$

52. $17^3 \div 15^2x^3$

EXTENSIONS

For the following exercises, simplify the given expression. Write answers with positive exponents.

53. $\left(\dfrac{3^2}{a^3}\right)^{-2}\left(\dfrac{a^4}{2^2}\right)^2$

54. $(6^2 - 24)^2 \div \left(\dfrac{x}{y}\right)^{-5}$

55. $\dfrac{m^2n^3}{a^2c^{-3}} \cdot \dfrac{a^{-7}n^{-2}}{m^2c^4}$

56. $\left(\dfrac{x^6y^3}{x^3y^{-3}} \cdot \dfrac{y^{-7}}{x^{-3}}\right)^{10}$

57. $\left(\dfrac{(ab^2c)^{-3}}{b^{-3}}\right)^2$

58. Avogadro's constant is used to calculate the number of particles in a mole. A mole is a basic unit in chemistry to measure the amount of a substance. The constant is 6.0221413×10^{23}. Write Avogadro's constant in standard notation.

59. Planck's constant is an important unit of measure in quantum physics. It describes the relationship between energy and frequency. The constant is written as $6.62606957 \times 10^{-34}$. Write Planck's constant in standard notation.

LEARNING OBJECTIVES

In this section, you will:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

1.3 RADICALS AND RATIONAL EXPRESSIONS

A hardware store sells 16-ft ladders and 24-ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in **Figure 1**, and use the Pythagorean Theorem.

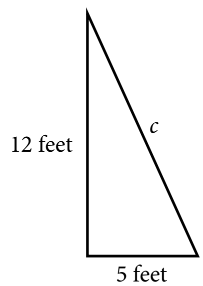


Figure 1

$$a^2 + b^2 = c^2$$
$$5^2 + 12^2 = c^2$$
$$169 = c^2$$

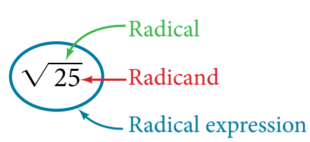
Now, we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other words, we need to find a square root. In this section, we will investigate methods of finding solutions to problems such as this one.

Evaluating Square Roots

When the square root of a number is squared, the result is the original number. Since $4^2 = 16$, the square root of 16 is 4. The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if a is a positive real number, then the square root of a is a number that, when multiplied by itself, gives a. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The **principal square root** is the nonnegative number that when multiplied by itself equals a. The square root obtained using a calculator is the principal square root.

The principal square root of a is written as $\sqrt{a}$. The symbol is called a **radical**, the term under the symbol is called the **radicand**, and the entire expression is called a **radical expression**.



principal square root

The **principal square root** of a is the nonnegative number that, when multiplied by itself, equals a. It is written as a **radical expression**, with a symbol called a **radical** over the term called the **radicand**: $\sqrt{a}$.

Q & A…

Does $\sqrt{25} = \pm 5$?

No. Although both 5^2 and $(-5)^2$ are 25, the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is $\sqrt{25} = 5$.

Example 1 Evaluating Square Roots

Evaluate each expression.

a. $\sqrt{100}$ **b.** $\sqrt{\sqrt{16}}$ **c.** $\sqrt{25 + 144}$ **d.** $\sqrt{49} - \sqrt{81}$

Solution

a. $\sqrt{100} = 10$ because $10^2 = 100$

b. $\sqrt{\sqrt{16}} = \sqrt{4} = 2$ because $4^2 = 16$ and $2^2 = 4$

c. $\sqrt{25 + 144} = \sqrt{169} = 13$ because $13^2 = 169$

d. $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$ because $7^2 = 49$ and $9^2 = 81$

Q & A…

For $\sqrt{25 + 144}$, can we find the square roots before adding?

No. $\sqrt{25} + \sqrt{144} = 5 + 12 = 17$. This is not equivalent to $\sqrt{25 + 144} = 13$. The order of operations requires us to add the terms in the radicand before finding the square root.

Try It #1

a. $\sqrt{225}$ **b.** $\sqrt{\sqrt{81}}$ **c.** $\sqrt{25 - 9}$ **d.** $\sqrt{36} + \sqrt{121}$

Using the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the *product rule for simplifying square roots*, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

the product rule for simplifying square roots

If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b.

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

How To…

Given a square root radical expression, use the product rule to simplify it.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

Example 2 Using the Product Rule to Simplify Square Roots

Simplify the radical expression.

a. $\sqrt{300}$ **b.** $\sqrt{162a^5b^4}$

Solution

a. $\sqrt{100 \cdot 3}$ Factor perfect square from radicand.

$\sqrt{100} \cdot \sqrt{3}$ Write radical expression as product of radical expressions.

$10\sqrt{3}$ Simplify.

b. $\sqrt{81a^4b^4 \cdot 2a}$ Factor perfect square from radicand.

$\sqrt{81a^4b^4} \cdot \sqrt{2a}$ Write radical expression as product of radical expressions.

$9a^2b^2\sqrt{2a}$ Simplify.

Try It #2

Simplify $\sqrt{50x^2y^3z}$.

How To...

Given the product of multiple radical expressions, use the product rule to combine them into one radical expression.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

Example 3 Using the Product Rule to Simplify the Product of Multiple Square Roots

Simplify the radical expression. $\sqrt{12} \cdot \sqrt{3}$

Solution

$\sqrt{12 \cdot 3}$ Express the product as a single radical expression.

$\sqrt{36}$ Simplify.

6

Try It #3

Simplify $\sqrt{50x} \cdot \sqrt{2x}$ assuming $x > 0$.

Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the *quotient rule for simplifying square roots*. It can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

the quotient rule for simplifying square roots

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b, where $b \neq 0$.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

How To...

Given a radical expression, use the quotient rule to simplify it.

1. Write the radical expression as the quotient of two radical expression.
2. Simplify the numerator and denominator.

Example 4 Using the Quotient Rule to Simplify Square Roots

Simplify the radical expression. $\sqrt{\frac{5}{36}}$

Solution

$\frac{\sqrt{5}}{\sqrt{36}}$ Write as quotient of two radical expressions.

$\frac{\sqrt{5}}{6}$ Simplify denominator.

Try It #4

Simplify $\sqrt{\dfrac{2x^2}{9y^4}}$.

Example 5 Using the Quotient Rule to Simplify an Expression with Two Square Roots

Simplify the radical expression. $\dfrac{\sqrt{234x^{11}y}}{\sqrt{26x^7y}}$

Solution

$\sqrt{\dfrac{234x^{11}y}{26x^7y}}$	Combine numerator and denominator into one radical expression.
$\sqrt{9x^4}$	Simplify fraction.
$3x^2$	Simplify square root.

Try It #5

Simplify $\dfrac{\sqrt{9a^5b^{14}}}{\sqrt{3a^4b^5}}$.

Adding and Subtracting Square Roots

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3\sqrt{2}$ is $4\sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3\sqrt{2}$, so $\sqrt{2} + \sqrt{18} = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$.

How To...

Given a radical expression requiring addition or subtraction of square roots, solve.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

Example 6 Adding Square Roots

Add $5\sqrt{12} + 2\sqrt{3}$.

Solution

We can rewrite $5\sqrt{12}$ as $5\sqrt{4 \cdot 3}$. According the product rule, this becomes $5\sqrt{4}\,\sqrt{3}$. The square root of $\sqrt{4}$ is 2, so the expression becomes $5(2)\sqrt{3}$, which is $10\sqrt{3}$. Now the terms have the same radicand so we can add.

$10\sqrt{3} + 2\sqrt{3} = 12\sqrt{3}$

Try It #6

Add $\sqrt{5} + 6\sqrt{20}$.

Example 7 Subtracting Square Roots

Subtract $20\sqrt{72a^3b^4c} - 14\sqrt{8a^3b^4c}$.

Solution

Rewrite each term so they have equal radicands.

$$
\begin{aligned}
20\sqrt{72a^3b^4c} &= 20\sqrt{9}\sqrt{4}\sqrt{2}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\
&= 20(3)(2)|a|b^2\sqrt{2ac} \\
&= 120|a|b^2\sqrt{2ac} \\
14\sqrt{8a^3b^4c} &= 14\sqrt{2}\sqrt{4}\sqrt{a}\sqrt{a^2}\sqrt{(b^2)^2}\sqrt{c} \\
&= 14(2)|a|b^2\sqrt{2ac} \\
&= 28|a|b^2\sqrt{2ac}
\end{aligned}
$$

Now the terms have the same radicand so we can subtract.

$$120|a|b^2\sqrt{2ac} - 28|a|b^2\sqrt{2ac} = 92|a|b^2\sqrt{2ac}$$

Try It #7

Subtract $3\sqrt{80x} - 4\sqrt{45x}$.

Rationalizing Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called *rationalizing the denominator.*

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the denominators of fractions, multiply by the form of 1 that will eliminate the radical.

For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b\sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a + b\sqrt{c}$, then the conjugate is $a - b\sqrt{c}$.

How To...

Given an expression with a single square root radical term in the denominator, rationalize the denominator.

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

Example 8 **Rationalizing a Denominator Containing a Single Term**

Write $\frac{2\sqrt{3}}{3\sqrt{10}}$ in simplest form.

Solution

The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$\frac{2\sqrt{3}}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$\frac{2\sqrt{30}}{30}$$

$$\frac{\sqrt{30}}{15}$$

Try It #8

Write $\dfrac{12\sqrt{3}}{\sqrt{2}}$ in simplest form.

How To…

Given an expression with a radical term and a constant in the denominator, rationalize the denominator.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

Example 9 Rationalizing a Denominator Containing Two Terms

Write $\dfrac{4}{1+\sqrt{5}}$ in simplest form.

Solution

Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\dfrac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$\frac{4}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$$\frac{4-4\sqrt{5}}{-4} \qquad \text{Use the distributive property.}$$

$$\sqrt{5}-1 \qquad \text{Simplify.}$$

Try It #9

Write $\dfrac{7}{2+\sqrt{3}}$ in simplest form.

Using Rational Roots

Although square roots are the most common rational roots, we can also find cuberoots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number that, when raised to a certain power, gives a certain number.

Understanding *n*th Roots

Suppose we know that $a^3 = 8$. We want to find what number raised to the 3rd power is equal to 8. Since $2^3 = 8$, we say that 2 is the cube root of 8.

The nth root of a is a number that, when raised to the nth power, gives a. For example, -3 is the 5th root of -243 because $(-3)^5 = -243$. If a is a real number with at least one nth root, then the **principal *n*th root** of a is the number with the same sign as a that, when raised to the nth power, equals a.

The principal nth root of a is written as $\sqrt[n]{a}$, where n is a positive integer greater than or equal to 2. In the radical expression, n is called the **index** of the radical.

principal nth root

If a is a real number with at least one nth root, then the **principal *n*th root** of a, written as $\sqrt[n]{a}$, is the number with the same sign as a that, when raised to the nth power, equals a. The **index** of the radical is n.

Example 10 Simplifying *n*th Roots

Simplify each of the following:

a. $\sqrt[5]{-32}$ **b.** $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$ **c.** $-\sqrt[3]{\dfrac{8x^6}{125}}$ **d.** $8\sqrt[4]{3} - \sqrt[4]{48}$

Solution

a. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

b. First, express the product as a single radical expression. $\sqrt[4]{4,096} = 8$ because $8^4 = 4,096$

c. $\dfrac{-\sqrt[3]{8x^6}}{\sqrt[3]{125}}$ Write as quotient of two radical expressions.

$\dfrac{-2x^2}{5}$ Simplify.

d. $8\sqrt[4]{3} - 2\sqrt[4]{3}$ Simplify to get equal radicands.

$6\sqrt[4]{3}$ Add.

Try It #10

Simplify.

a. $\sqrt[3]{-216}$ **b.** $\dfrac{3\sqrt[4]{80}}{\sqrt[4]{5}}$ **c.** $6\sqrt[3]{9,000} + 7\sqrt[3]{576}$

Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index n is even, then a cannot be negative.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

rational exponents

Rational exponents are another way to express principal nth roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

How To…

Given an expression with a rational exponent, write the expression as a radical.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Example 11 Writing Rational Exponents as Radicals

Write $343^{\frac{2}{3}}$ as a radical. Simplify.

Solution

The 2 tells us the power and the 3 tells us the root.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = \sqrt[3]{343^2}$$

We know that $\sqrt[3]{343} = 7$ because $7^3 = 343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.

$$343^{\frac{2}{3}} = \left(\sqrt[3]{343}\right)^2 = 7^2 = 49$$

Try It #11

Write $9^{\frac{5}{2}}$ as a radical. Simplify.

Example 12 Writing Radicals as Rational Exponents

Write $\dfrac{4}{\sqrt[7]{a^2}}$ using a rational exponent.

Solution

The power is 2 and the root is 7, so the rational exponent will be $\frac{2}{7}$. We get $\dfrac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\dfrac{4}{\sqrt[7]{a^2}} = 4a^{\frac{-2}{7}}$.

Try It #12

Write $x\sqrt{(5y)^9}$ using a rational exponent.

Example 13 Simplifying Rational Exponents

Simplify:

a. $5\left(2x^{\frac{3}{4}}\right)\left(3x^{\frac{1}{5}}\right)$ **b.** $\left(\dfrac{16}{9}\right)^{-\frac{1}{2}}$

Solution

a. $30x^{\frac{3}{4}}x^{\frac{1}{5}}$	Multiply the coefficient.
$30x^{\frac{3}{4}+\frac{1}{5}}$	Use properties of exponents.
$30x^{\frac{19}{20}}$	Simplify.
b. $\left(\dfrac{9}{16}\right)^{\frac{1}{2}}$	Use definition of negative exponents.
$\sqrt{\dfrac{9}{16}}$	Rewrite as a radical.
$\dfrac{\sqrt{9}}{\sqrt{16}}$	Use the quotient rule.
$\dfrac{3}{4}$	Simplify.

Try It #13

Simplify $8x^{\frac{1}{3}}\left(14x^{\frac{6}{5}}\right)$.

Access these online resources for additional instruction and practice with radicals and rational exponents.

- Radicals (http://openstaxcollege.org/l/introradical)
- Rational Exponents (http://openstaxcollege.org/l/rationexpon)
- Simplify Radicals (http://openstaxcollege.org/l/simpradical)
- Rationalize Denominator (http://openstaxcollege.org/l/rationdenom)

1.3 SECTION EXERCISES

VERBAL

1. What does it mean when a radical does not have an index? Is the expression equal to the radicand? Explain.

2. Where would radicals come in the order of operations? Explain why.

3. Every number will have two square roots. What is the principal square root?

4. Can a radical with a negative radicand have a real square root? Why or why not?

NUMERIC

For the following exercises, simplify each expression.

5. $\sqrt{256}$

6. $\sqrt{\sqrt{256}}$

7. $\sqrt{4(9+16)}$

8. $\sqrt{289}-\sqrt{121}$

9. $\sqrt{196}$

10. $\sqrt{1}$

11. $\sqrt{98}$

12. $\sqrt{\frac{27}{64}}$

13. $\sqrt{\frac{81}{5}}$

14. $\sqrt{800}$

15. $\sqrt{169}+\sqrt{144}$

16. $\sqrt{\frac{8}{50}}$

17. $\frac{18}{\sqrt{162}}$

18. $\sqrt{192}$

19. $14\sqrt{6}-6\sqrt{24}$

20. $15\sqrt{5}+7\sqrt{45}$

21. $\sqrt{150}$

22. $\sqrt{\frac{96}{100}}$

23. $(\sqrt{42})(\sqrt{30})$

24. $12\sqrt{3}-4\sqrt{75}$

25. $\sqrt{\frac{4}{225}}$

26. $\sqrt{\frac{405}{324}}$

27. $\sqrt{\frac{360}{361}}$

28. $\frac{5}{1+\sqrt{3}}$

29. $\frac{8}{1-\sqrt{17}}$

30. $\sqrt[4]{16}$

31. $\sqrt[3]{128}+3\sqrt[3]{2}$

32. $\sqrt[5]{\frac{-32}{243}}$

33. $\frac{15\sqrt[4]{125}}{\sqrt[4]{5}}$

34. $3\sqrt[3]{-432}+\sqrt[3]{16}$

ALGEBRAIC

For the following exercises, simplify each expression.

35. $\sqrt{400x^4}$

36. $\sqrt{4y^2}$

37. $\sqrt{49p}$

38. $(144p^2q^6)^{\frac{1}{2}}$

39. $m^{\frac{5}{2}}\sqrt{289}$

40. $9\sqrt{3m^2}+\sqrt{27}$

41. $3\sqrt{ab^2}-b\sqrt{a}$

42. $\frac{4\sqrt{2n}}{\sqrt{16n^4}}$

43. $\sqrt{\frac{225x^3}{49x}}$

44. $3\sqrt{44z}+\sqrt{99z}$

45. $\sqrt{50y^8}$

46. $\sqrt{490bc^2}$

47. $\sqrt{\frac{32}{14d}}$

48. $q^{\frac{3}{2}}\sqrt{63p}$

49. $\frac{\sqrt{8}}{1-\sqrt{3x}}$

50. $\sqrt{\frac{20}{121d^4}}$

51. $w^{\frac{3}{2}}\sqrt{32}-w^{\frac{3}{2}}\sqrt{50}$

52. $\sqrt{108x^4}+\sqrt{27x^4}$

53. $\frac{\sqrt{12x}}{2+2\sqrt{3}}$

54. $\sqrt{147k^3}$

55. $\sqrt{125n^{10}}$

56. $\sqrt{\frac{42q}{36q^3}}$

57. $\sqrt{\frac{81m}{361m^2}}$

58. $\sqrt{72c}-2\sqrt{2c}$

59. $\sqrt{\dfrac{144}{324d^2}}$

60. $\sqrt[3]{24x^6} + \sqrt[3]{81x^6}$

61. $\sqrt[4]{\dfrac{162x^6}{16x^4}}$

62. $\sqrt[3]{64y}$

63. $\sqrt[3]{128z^3} - \sqrt[3]{-16z^3}$

64. $\sqrt[5]{1{,}024c^{10}}$

REAL-WORLD APPLICATIONS

65. A guy wire for a suspension bridge runs from the ground diagonally to the top of the closest pylon to make a triangle. We can use the Pythagorean Theorem to find the length of guy wire needed. The square of the distance between the wire on the ground and the pylon on the ground is 90,000 feet. The square of the height of the pylon is 160,000 feet. So the length of the guy wire can be found by evaluating $\sqrt{90{,}000 + 160{,}000}$. What is the length of the guy wire?

66. A car accelerates at a rate of $6 - \dfrac{\sqrt{4}}{\sqrt{t}}$ m/s^2 where t is the time in seconds after the car moves from rest. Simplify the expression.

EXTENSIONS

For the following exercises, simplify each expression.

67. $\dfrac{\sqrt{8} - \sqrt{16}}{4 - \sqrt{2}} - 2^{\frac{1}{2}}$

68. $\dfrac{4^{\frac{3}{2}} - 16^{\frac{3}{2}}}{8^{\frac{1}{3}}}$

69. $\dfrac{\sqrt{mn^3}}{a^2\sqrt{c^{-3}}} \cdot \dfrac{a^{-7}n^{-2}}{\sqrt{m^2c^4}}$

70. $\dfrac{a}{a - \sqrt{c}}$

71. $\dfrac{x\sqrt{64y} + 4\sqrt{y}}{\sqrt{128y}}$

72. $\left(\dfrac{\sqrt{250x^2}}{\sqrt{100b^3}}\right)\left(\dfrac{7\sqrt{b}}{\sqrt{125x}}\right)$

73. $\sqrt{\dfrac{\sqrt[3]{64} + \sqrt[4]{256}}{\sqrt{64} + \sqrt{256}}}$

LEARNING OBJECTIVES

In this section, you will:

- Identify the degree and leading coefficient of polynomials.
- Add and subtract polynomials.
- Multiply polynomials.
- Use FOIL to multiply binomials.
- Perform operations with polynomials of several variables.

1.4 POLYNOMIALS

Earl is building a doghouse, whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house, shown in **Figure 1**, we can create an expression that combines several variable terms, allowing us to solve this problem and others like it.

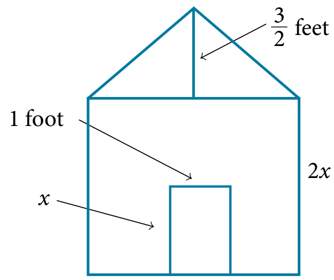


Figure 1

First find the area of the square in square feet.

$$\begin{aligned} A &= s^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

Next find the area of the rectangular door in square feet.

$$\begin{aligned} A &= lw \\ &= x \cdot 1 \\ &= x \end{aligned}$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle, and then subtracting the area of the rectangle. When we do this, we get $4x^2 + \frac{3}{2}x - x$ ft^2, or $4x^2 + \frac{1}{2}x$ ft^2.

In this section, we will examine expressions such as this one, which combine several variable terms.

Identifying the Degree and Leading Coefficient of Polynomials

The formula just found is an example of a **polynomial**, which is a sum of or difference of terms, each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as 384π, is known as a **coefficient**. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_i x^i$, such as $384\pi w$, is a **term of a polynomial**. If a term does not contain a variable, it is called a *constant*.

A polynomial containing only one term, such as $5x^4$, is called a **monomial**. A polynomial containing two terms, such as $2x - 9$, is called a **binomial**. A polynomial containing three terms, such as $-3x^2 + 8x - 7$, is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers are descending, we say that it is in standard form.

Leading coefficient Degree

$$\underbrace{a_n x^n}_{\text{Leading term}} + \ldots + a_2 x^2 + a_1 x + a_0$$

polynomials

A **polynomial** is an expression that can be written in the form

$$a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$$

Each real number a_i is called a **coefficient**. The number a_0 that is not multiplied by a variable is called a *constant*. Each product $a_i x^i$ is a **term of a polynomial**. The highest power of the variable that occurs in the polynomial is called the **degree** of a polynomial. The **leading term** is the term with the highest power, and its coefficient is called the **leading coefficient**.

How To...

Given a polynomial expression, identify the degree and leading coefficient.

1. Find the highest power of x to determine the degree.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 1 Identifying the Degree and Leading Coefficient of a Polynomial

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

a. $3 + 2x^2 - 4x^3$ **b.** $5t^5 - 2t^3 + 7t$ **c.** $6p - p^3 - 2$

Solution

a. The highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

b. The highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

c. The highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$, The leading coefficient is the coefficient of that term, -1.

Try It #1

Identify the degree, leading term, and leading coefficient of the polynomial $4x^2 - x^6 + 2x - 6$.

Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5x^2$ and $-2x^2$ are like terms, and can be added to get $3x^2$, but $3x$ and $3x^2$ are not like terms, and therefore cannot be added.

How To…

Given multiple polynomials, add or subtract them to simplify the expressions.

1. Combine like terms.
2. Simplify and write in standard form.

Example 2 **Adding Polynomials**

Find the sum.

$(12x^2 + 9x - 21) + (4x^3 + 8x^2 - 5x + 20)$

Solution

$4x^3 + (12x^2 + 8x^2) + (9x - 5x) + (-21 + 20)$	Combine like terms.
$4x^3 + 20x^2 + 4x - 1$	Simplify.

Analysis *We can check our answers to these types of problems using a graphing calculator. To check, graph the problem as given along with the simplified answer. The two graphs should be equivalent. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.*

Try It #2

Find the sum.

$(2x^3 + 5x^2 - x + 1) + (2x^2 - 3x - 4)$

Example 3 **Subtracting Polynomials**

Find the difference.

$(7x^4 - x^2 + 6x + 1) - (5x^3 - 2x^2 + 3x + 2)$

Solution

$7x^4 - 5x^3 + (-x^2 + 2x^2) + (6x - 3x) + (1 - 2)$	Combine like terms.
$7x^4 - 5x^3 + x^2 + 3x - 1$	Simplify.

Analysis *Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.*

Try It #3

Find the difference.

$(-7x^3 - 7x^2 + 6x - 2) - (4x^3 - 6x^2 - x + 7)$

Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial, we use the distributive property. The number must be distributed to each term of the polynomial. We can distribute the 2 in $2(x + 7)$ to obtain the equivalent expression $2x + 14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

How To…

Given the multiplication of two polynomials, use the distributive property to simplify the expression.

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

Example 4 Multiplying Polynomials Using the Distributive Property

Find the product.

$(2x + 1)(3x^2 - x + 4)$

Solution

$2x(3x^2 - x + 4) + 1(3x^2 - x + 4)$	Use the distributive property.
$(6x^3 - 2x^2 + 8x) + (3x^2 - x + 4)$	Multiply.
$6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4$	Combine like terms.
$6x^3 + x^2 + 7x + 4$	Simplify.

Analysis *We can use a table to keep track of our work, as shown in **Table 1**. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.*

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	4

Table 1

Try It #4

Find the product.

$(3x + 2)(x^3 - 4x^2 + 7)$

Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.

First terms Last terms

$$(ax + b)(cx + d) = acx^2 + adx + bcx + bd$$

Inner terms

Outer terms

The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial, and then combining like terms.

How To…

Given two binomials, use FOIL to simplify the expression.

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

Example 5 Using FOIL to Multiply Binomials

Use FOIL to find the product.

$(2x - 10)(3x + 3)$

Solution

Find the product of the first terms.

$2x - 18 \quad 3x + 3 \qquad\qquad 2x \cdot 3x = 6x^2$

Find the product of the outer terms.

$2x - 18 \quad 3x + 3 \qquad\qquad 2x \cdot 3 = 6x$

Find the product of the inner terms.

$2x - 18 \quad 3x + 3 \qquad\qquad -18 \cdot 3x = -54x$

Find the product of the last terms.

$2x - 18 \quad 3x + 3 \qquad\qquad -18 \cdot 3 = -54$

$6x^2 + 6x - 54x - 54$	Add the products.
$6x^2 + (6x - 54x) - 54$	Combine like terms.
$6x^2 - 48x - 54$	Simplify.

Try It #5

Use FOIL to find the product.

$(x + 7)(3x - 5)$

Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a **perfect square trinomial**. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier and faster. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$(x + 5)^2 = x^2 + 10x + 25$$
$$(x - 3)^2 = x^2 - 6x + 9$$
$$(4x - 1)^2 = 4x^2 - 8x + 1$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

perfect square trinomials

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$(x + a)^2 = (x + a)(x + a) = x^2 + 2ax + a^2$$

How To…

Given a binomial, square it using the formula for perfect square trinomials.

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

Example 6 Expanding Perfect Squares

Expand $(3x - 8)^2$.

Solution Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$(3x)^2 - 2(3x)(8) + (-8)^2$$

Simplify.

$$9x^2 - 48x + 64$$

Try It #6

Expand $(4x - 1)^2$.

Difference of Squares

Another special product is called the **difference of squares**, which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x + 1)(x - 1)$ using the FOIL method.

$$\begin{aligned}(x + 1)(x - 1) &= x^2 - x + x - 1 \\ &= x^2 - 1\end{aligned}$$

The middle term drops out, resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.

$$\begin{aligned}(x + 5)(x - 5) &= x^2 - 25 \\ (x + 11)(x - 11) &= x^2 - 121 \\ (2x + 3)(2x - 3) &= 4x^2 - 9\end{aligned}$$

Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

Q & A…

Is there a special form for the sum of squares?

No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

difference of squares

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$(a + b)(a - b) = a^2 - b^2$$

How To…

Given a binomial multiplied by a binomial with the same terms but the opposite sign, find the difference of squares.

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

Example 7 Multiplying Binomials Resulting in a Difference of Squares

Multiply $(9x + 4)(9x - 4)$.

Solution Square the first term to get $(9x)^2 = 81x^2$. Square the last term to get $4^2 = 16$. Subtract the square of the last term from the square of the first term to find the product of $81x^2 - 16$.

Try It #7

Multiply $(2x + 7)(2x - 7)$.

Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:

$(a + 2b)(4a - b - c)$	
$a(4a - b - c) + 2b(4a - b - c)$	Use the distributive property.
$4a^2 - ab - ac + 8ab - 2b^2 - 2bc$	Multiply.
$4a^2 + (- ab + 8ab) - ac - 2b^2 - 2bc$	Combine like terms.
$4a^2 + 7ab - ac - 2bc - 2b^2$	Simplify.

Example 8 Multiplying Polynomials Containing Several Variables

Multiply $(x + 4)(3x - 2y + 5)$.

Solution Follow the same steps that we used to multiply polynomials containing only one variable.

$x(3x - 2y + 5) + 4(3x - 2y + 5)$	Use the distributive property.
$3x^2 - 2xy + 5x + 12x - 8y + 20$	Multiply.
$3x^2 - 2xy + (5x + 12x) - 8y + 20$	Combine like terms.
$3x^2 - 2xy + 17x - 8y + 20$	Simplify.

Try It #8

Multiply $(3x - 1)(2x + 7y - 9)$.

Access these online resources for additional instruction and practice with polynomials.

- Adding and Subtracting Polynomials (http://openstaxcollege.org/l/addsubpoly)
- Multiplying Polynomials (http://openstaxcollege.org/l/multiplpoly)
- Special Products of Polynomials (http://openstaxcollege.org/l/specialpolyprod)

1.4 SECTION EXERCISES

VERBAL

1. Evaluate the following statement: The degree of a polynomial in standard form is the exponent of the leading term. Explain why the statement is true or false.

2. Many times, multiplying two binomials with two variables results in a trinomial. This is not the case when there is a difference of two squares. Explain why the product in this case is also a binomial.

3. You can multiply polynomials with any number of terms and any number of variables using four basic steps over and over until you reach the expanded polynomial. What are the four steps?

4. State whether the following statement is true and explain why or why not: A trinomial is always a higher degree than a monomial.

ALGEBRAIC

For the following exercises, identify the degree of the polynomial.

5. $7x - 2x^2 + 13$
6. $14m^3 + m^2 - 16m + 8$
7. $-625a^8 + 16b^4$
8. $200p - 30p^2m + 40m^3$
9. $x^2 + 4x + 4$
10. $6y^4 - y^5 + 3y - 4$

For the following exercises, find the sum or difference.

11. $(12x^2 + 3x) -(8x^2 -19)$
12. $(4z^3 + 8z^2 - z) + (-2z^2 + z + 6)$
13. $(6w^2 + 24w + 24) - (3w - 6w + 3)$
14. $(7a^3 + 6a^2 - 4a - 13) + (-3a^3 - 4a^2 + 6a + 17)$
15. $(11b^4 - 6b^3 + 18b^2 - 4b + 8) - (3b^3 + 6b^2 + 3b)$
16. $(49p^2 - 25) + (16p^4 - 32p^2 + 16)$

For the following exercises, find the product.

17. $(4x + 2)\ (6x - 4)$
18. $(14c^2 + 4c)\ (2c^2 - 3c)$
19. $(6b^2 - 6)\ (4b^2 - 4)$
20. $(3d - 5)(2d + 9)$
21. $(9v - 11)\ (11v - 9)$
22. $(4t^2 + 7t)\ (-3t^2 + 4)$
23. $(8n - 4)\ (n^2 + 9)$

For the following exercises, expand the binomial.

24. $(4x + 5)^2$
25. $(3y - 7)^2$
26. $(12 - 4x)^2$
27. $(4p + 9)^2$
28. $(2m - 3)^2$
29. $(3y - 6)^2$
30. $(9b + 1)^2$

For the following exercises, multiply the binomials.

31. $(4c + 1)(4c - 1)$
32. $(9a - 4)(9a + 4)$
33. $(15n - 6)(15n + 6)$
34. $(25b + 2)(25b - 2)$
35. $(4 + 4m)(4 - 4m)$
36. $(14p + 7)(14p - 7)$
37. $(11q - 10)(11q + 10)$

For the following exercises, find the sum or difference.

38. $(2x^2 + 2x + 1)\ (4x - 1)$
39. $(4t^2 + t - 7)\ (4t^2 - 1)$
40. $(x - 1)\ (x^2 - 2x + 1)$
41. $(y - 2)\ (y^2 - 4y - 9)$
42. $(6k - 5)\ (6k^2 + 5k - 1)$
43. $(3p^2 + 2p - 10)\ (p - 1)$
44. $(4m - 13)\ (2m^2 - 7m + 9)$
45. $(a + b)\ (a - b)$
46. $(4x - 6y)\ (6x - 4y)$
47. $(4t - 5u)^2$
48. $(9m + 4n - 1)\ (2m + 8)$
49. $(4t - x)(t - x + 1)$
50. $(b^2 - 1)\ (a^2 + 2ab + b^2)$
51. $(4r - d)(6r + 7d)$
52. $(x + y)\ (x^2 - xy + y^2)$

REAL-WORLD APPLICATIONS

53. A developer wants to purchase a plot of land to build a house. The area of the plot can be described by the following expression: $(4x + 1)(8x - 3)$ where x is measured in meters. Multiply the binomials to find the area of the plot in standard form.

54. A prospective buyer wants to know how much grain a specific silo can hold. The area of the floor of the silo is $(2x + 9)^2$. The height of the silo is $10x + 10$, where x is measured in feet. Expand the square and multiply by the height to find the expression that shows how much grain the silo can hold.

EXTENSIONS

For the following exercises, perform the given operations.

55. $(4t - 7)^2\ (2t + 1) - (4t^2 + 2t + 11)$
56. $(3b + 6)(3b - 6)\ (9b^2 - 36)$
57. $(a^2 + 4ac + 4c^2)\ (a^2 - 4c^2)$

LEARNING OBJECTIVES

In this section, you will:

- Factor the greatest common factor of a polynomial.
- Factor a trinomial.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor the sum and difference of cubes.
- Factor expressions using fractional or negative exponents.

1.5 FACTORING POLYNOMIALS

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in **Figure 1**.

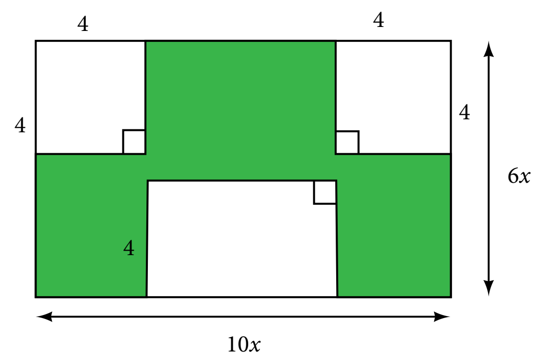


Figure 1

The area of the entire region can be found using the formula for the area of a rectangle.

$$\begin{aligned} A &= lw \\ &= 10x \cdot 6x \\ &= 60x^2 \text{ units}^2 \end{aligned}$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A = s^2 = 4^2 = 16$ units^2. The other rectangular region has one side of length $10x - 8$ and one side of length 4, giving an area of $A = lw = 4(10x - 8) = 40x - 32$ units^2. So the region that must be subtracted has an area of $2(16) + 40x - 32 = 40x$ units^2.

The area of the region that requires grass seed is found by subtracting $60x^2 - 40x$ units^2. This area can also be expressed in factored form as $20x(3x - 2)$ units^2. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

Factoring the Greatest Common Factor of a Polynomial

When we study fractions, we learn that the **greatest common factor (GCF)** of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 The GCF of polynomials works the same way: $4x$ is the GCF of $16x$ and $20x^2$ because it is the largest polynomial that divides evenly into both $16x$ and $20x^2$.

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

greatest common factor

The **greatest common factor** (GCF) of polynomials is the largest polynomial that divides evenly into the polynomials.

How To...

Given a polynomial expression, factor out the greatest common factor.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

Example 1 Factoring the Greatest Common Factor

Factor $6x^3y^3 + 45x^2y^2 + 21xy$.

Solution First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of x^3, x^2, and x is x. (Note that the GCF of a set of expressions in the form x^n will always be the exponent of lowest degree.) And the GCF of y^3, y^2, and y is y. Combine these to find the GCF of the polynomial, $3xy$.

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3xy(2x^2y^2) = 6x^3y^3$, $3xy(15xy) = 45x^2y^2$, and $3xy(7) = 21xy$.

Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$(3xy)(2x^2y^2 + 15xy + 7)$$

Analysis *After factoring, we can check our work by multiplying. Use the distributive property to confirm that* $(3xy)(2x^2y^2 + 15xy + 7) = 6x^3y^3 + 45x^2y^2 + 21xy$.

Try It #1

Factor $x(b^2 - a) + 6(b^2 - a)$ by pulling out the GCF.

Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that polynomial expressions can be factored. The polynomial $x^2 + 5x + 6$ has a GCF of 1, but it can be written as the product of the factors $(x + 2)$ and $(x + 3)$.

Trinomials of the form $x^2 + bx + c$ can be factored by finding two numbers with a product of c and a sum of b. The trinomial $x^2 + 10x + 16$, for example, can be factored using the numbers 2 and 8 because the product of those numbers is 16 and their sum is 10. The trinomial can be rewritten as the product of $(x + 2)$ and $(x + 8)$.

***factoring a trinomial with leading coefficient* 1**

A trinomial of the form $x^2 + bx + c$ can be written in factored form as $(x + p)(x + q)$ where $pq = c$ and $p + q = b$.

Q & A...

Can every trinomial be factored as a **product of binomials?**

No. Some polynomials cannot be factored. These polynomials are said to be prime.

How To...

Given a trinomial in the form $x^2 + bx + c$, factor it.

1. List factors of c.
2. Find p and q, a pair of factors of c with a sum of b.
3. Write the factored expression $(x + p)(x + q)$.

Example 2 Factoring a Trinomial with Leading Coefficient 1

Factor $x^2 + 2x - 15$.

Solution We have a trinomial with leading coefficient 1, $b = 2$, and $c = -15$. We need to find two numbers with a product of -15 and a sum of 2. In **Table 1**, we list factors until we find a pair with the desired sum.

Factors of -15	Sum of Factors
$1, -15$	-14
$-1, 15$	14
$3, -5$	-2
$-3, 5$	2

Table 1

Now that we have identified p and q as -3 and 5, write the factored form as $(x - 3)(x + 5)$.

Analysis *We can check our work by multiplying. Use FOIL to confirm that* $(x - 3)(x + 5) = x^2 + 2x - 15$.

Q & A...

Does the order of the factors matter?

No. Multiplication is commutative, so the order of the factors does not matter.

Try It #2

Factor $x^2 - 7x + 6$.

Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2x^2 + 5x + 3$ can be rewritten as $(2x + 3)(x + 1)$ using this process. We begin by rewriting the original expression as $2x^2 + 2x + 3x + 3$ and then factor each portion of the expression to obtain $2x(x + 1) + 3(x + 1)$. We then pull out the GCF of $(x + 1)$ to find the factored expression.

factor by grouping

To factor a trinomial in the form $ax^2 + bx + c$ by grouping, we find two numbers with a product of ac and a sum of b. We use these numbers to divide the x term into the sum of two terms and factor each portion of the expression separately, then factor out the GCF of the entire expression.

How To...

Given a trinomial in the form $ax^2 + bx + c$, factor by grouping.

1. List factors of ac.
2. Find p and q, a pair of factors of ac with a sum of b.
3. Rewrite the original expression as $ax^2 + px + qx + c$.
4. Pull out the GCF of $ax^2 + px$.
5. Pull out the GCF of $qx + c$.
6. Factor out the GCF of the expression.

Example 3 Factoring a Trinomial by Grouping

Factor $5x^2 + 7x - 6$ by grouping.

Solution We have a trinomial with $a = 5$, $b = 7$, and $c = -6$. First, determine $ac = -30$. We need to find two numbers with a product of -30 and a sum of 7. In **Table 2**, we list factors until we find a pair with the desired sum.

Factors of -30	Sum of Factors
$1, -30$	-29
$-1, 30$	29
$2, -15$	-13
$-2, 15$	13
$3, -10$	-7
$-3, 10$	7

Table 2

So $p = -3$ and $q = 10$.

$5x^2 - 3x + 10x - 6$ Rewrite the original expression as $ax^2 + px + qx + c$.

$x(5x - 3) + 2(5x - 3)$ Factor out the GCF of each part.

$(5x - 3)(x + 2)$ Factor out the GCF of the expression.

Analysis *We can check our work by multiplying. Use FOIL to confirm that* $(5x - 3)(x + 2) = 5x^2 + 7x - 6$.

Try It #3

Factor. **a.** $2x^2 + 9x + 9$ **b.** $6x^2 + x - 1$

Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

We can use this equation to factor any perfect square trinomial.

perfect square trinomials

A perfect square trinomial can be written as the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$

How To...

Given a perfect square trinomial, factor it into the square of a binomial.

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of ab.
3. Write the factored form as $(a + b)^2$.

Example 4 Factoring a Perfect Square Trinomial

Factor $25x^2 + 20x + 4$.

Solution Notice that $25x^2$ and 4 are perfect squares because $25x^2 = (5x)^2$ and $4 = 2^2$. Then check to see if the middle term is twice the product of $5x$ and 2. The middle term is, indeed, twice the product: $2(5x)(2) = 20x$.

Therefore, the trinomial is a perfect square trinomial and can be written as $(5x + 2)^2$.

Try It #4

Factor $49x^2 - 14x + 1$.

Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2 - b^2 = (a + b)(a - b)$$

We can use this equation to factor any differences of squares.

differences of squares
A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a + b)(a - b)$$

How To...
Given a difference of squares, factor it into binomials.

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a + b)(a - b)$.

Example 5 Factoring a Difference of Squares

Factor $9x^2 - 25$.

Solution Notice that $9x^2$ and 25 are perfect squares because $9x^2 = (3x)^2$ and $25 = 5^2$. The polynomial represents a difference of squares and can be rewritten as $(3x + 5)(3x - 5)$.

Try It #5

Factor $81y^2 - 100$.

Q & A...
Is there a formula to factor the sum of squares?

No. A sum of squares cannot be factored.

Factoring the Sum and Difference of Cubes

Now, we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial, but with different signs.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: **S**ame **O**pposite **A**lways **P**ositive. For example, consider the following example.

$$x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$$

The sign of the first 2 is the *same* as the sign between $x^3 - 2^3$. The sign of the $2x$ term is *opposite* the sign between $x^3 - 2^3$. And the sign of the last term, 4, is *always positive*.

sum and difference of cubes

We can factor the sum of two cubes as

$$a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$$

We can factor the difference of two cubes as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

How To…

Given a sum of cubes or difference of cubes, factor it.

1. Confirm that the first and last term are cubes, $a^3 + b^3$ or $a^3 - b^3$.
2. For a sum of cubes, write the factored form as $(a + b)(a^2 - ab + b^2)$. For a difference of cubes, write the factored form as $(a - b)(a^2 + ab + b^2)$.

Example 6 Factoring a Sum of Cubes

Factor $x^3 + 512$.

Solution Notice that x^3 and 512 are cubes because $8^3 = 512$. Rewrite the sum of cubes as $(x + 8)(x^2 - 8x + 64)$.

Analysis *After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.*

Try It #6

Factor the sum of cubes: $216a^3 + b^3$.

Example 7 Factoring a Difference of Cubes

Factor $8x^3 - 125$.

Solution Notice that $8x^3$ and 125 are cubes because $8x^3 = (2x)^3$ and $125 = 5^3$. Write the difference of cubes as $(2x - 5)(4x^2 + 10x + 25)$.

Analysis *Just as with the sum of cubes, we will not be able to further factor the trinomial portion.*

Try It #7

Factor the difference of cubes: $1{,}000x^3 - 1$.

Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2x^{\frac{1}{4}} + 5x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}\left(2 + 5x^{\frac{1}{2}}\right)$.

Example 8 **Factoring an Expression with Fractional or Negative Exponents**

Factor $3x(x + 2)^{-\frac{1}{3}} + 4(x + 2)^{\frac{2}{3}}$.

Solution Factor out the term with the lowest value of the exponent. In this case, that would be $(x + 2)^{-\frac{1}{3}}$.

$$(x + 2)^{-\frac{1}{3}}(3x + 4(x + 2)) \qquad \text{Factor out the GCF.}$$

$$(x + 2)^{-\frac{1}{3}}(3x + 4x + 8) \qquad \text{Simplify.}$$

$$(x + 2)^{-\frac{1}{3}}(7x + 8)$$

Try It #8

Factor $2(5a - 1)^{\frac{3}{4}} + 7a(5a - 1)^{-\frac{1}{4}}$.

Access these online resources for additional instruction and practice with factoring polynomials.

- Identify GCF (http://openstaxcollege.org/l/findgcftofact)
- Factor Trinomials when *a* Equals 1 (http://openstaxcollege.org/l/facttrinom1)
- Factor Trinomials when *a* is not equal to 1 (http://openstaxcollege.org/l/facttrinom2)
- Factor Sum or Difference of Cubes (http://openstaxcollege.org/l/sumdifcube)

1.5 SECTION EXERCISES

VERBAL

1. If the terms of a polynomial do not have a GCF, does that mean it is not factorable? Explain.

2. A polynomial is factorable, but it is not a perfect square trinomial or a difference of two squares. Can you factor the polynomial without finding the GCF?

3. How do you factor by grouping?

ALGEBRAIC

For the following exercises, find the greatest common factor.

4. $14x + 4xy - 18xy^2$

5. $49mb^2 - 35m^2ba + 77ma^2$

6. $30x^3y - 45x^2y^2 + 135xy^3$

7. $200p^3m^3 - 30p^2m^3 + 40m^3$

8. $36j^4k^2 - 18j^3k^3 + 54j^2k^4$

9. $6y^4 - 2y^3 + 3y^2 - y$

For the following exercises, factor by grouping.

10. $6x^2 + 5x - 4$

11. $2a^2 + 9a - 18$

12. $6c^2 + 41c + 63$

13. $6n^2 - 19n - 11$

14. $20w^2 - 47w + 24$

15. $2p^2 - 5p - 7$

For the following exercises, factor the polynomial.

16. $7x^2 + 48x - 7$

17. $10h^2 - 9h - 9$

18. $2b^2 - 25b - 247$

19. $9d^2 - 73d + 8$

20. $90v^2 - 181v + 90$

21. $12t^2 + t - 13$

22. $2n^2 - n - 15$

23. $16x^2 - 100$

24. $25y^2 - 196$

25. $121p^2 - 169$

26. $4m^2 - 9$

27. $361d^2 - 81$

28. $324x^2 - 121$

29. $144b^2 - 25c^2$

30. $16a^2 - 8a + 1$

31. $49n^2 + 168n + 144$

32. $121x^2 - 88x + 16$

33. $225y^2 + 120y + 16$

34. $m^2 - 20m + 100$

35. $m^2 - 20m + 100$

36. $36q^2 + 60q + 25$

For the following exercises, factor the polynomials.

37. $x^3 + 216$

38. $27y^3 - 8$

39. $125a^3 + 343$

40. $b^3 - 8d^3$

41. $64x^3 - 125$

42. $729q^3 + 1331$

43. $125r^3 + 1{,}728s^3$

44. $4x(x-1)^{-\frac{2}{3}} + 3(x-1)^{\frac{1}{3}}$

45. $3c(2c+3)^{-\frac{1}{4}} - 5(2c+3)^{\frac{3}{4}}$

46. $3t(10t+3)^{\frac{1}{3}} + 7(10t+3)^{\frac{4}{3}}$

47. $14x(x+2)^{-\frac{2}{5}} + 5(x+2)^{\frac{3}{5}}$

48. $9y(3y-13)^{\frac{1}{5}} - 2(3y-13)^{\frac{6}{5}}$

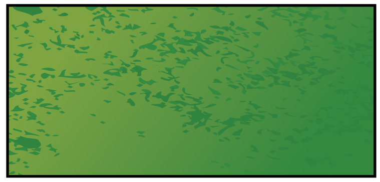
49. $5z(2z-9)^{-\frac{3}{2}} + 11(2z-9)^{-\frac{1}{2}}$

50. $6d(2d+3)^{-\frac{1}{6}} + 5(2d+3)^{\frac{5}{6}}$

REAL-WORLD APPLICATIONS

For the following exercises, consider this scenario:

Charlotte has appointed a chairperson to lead a city beautification project. The first act is to install statues and fountains in one of the city's parks. The park is a rectangle with an area of $98x^2 + 105x - 27$ m^2, as shown in the following figure. The length and width of the park are perfect factors of the area.



$l \times w = 98x^2 + 105x - 27$

51. Factor by grouping to find the length and width of the park.

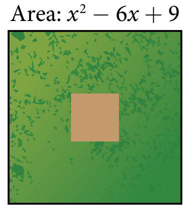
52. A statue is to be placed in the center of the park. The area of the base of the statue is $4x^2 + 12x + 9$ m^2. Factor the area to find the lengths of the sides of the statue.

53. At the northwest corner of the park, the city is going to install a fountain. The area of the base of the fountain is $9x^2 - 25$ m^2. Factor the area to find the lengths of the sides of the fountain.

For the following exercise, consider the following scenario:

A school is installing a flagpole in the central plaza. The plaza is a square with side length 100 yd as shown in the figure below. The flagpole will take up a square plot with area $x^2 - 6x + 9$ yd^2.

Area: $x^2 - 6x + 9$



100 yards

100 yards

54. Find the length of the base of the flagpole by factoring.

EXTENSIONS

For the following exercises, factor the polynomials completely.

55. $16x^4 - 200x^2 + 625$

56. $81y^4 - 256$

57. $16z^4 - 2{,}401a^4$

58. $5x(3x + 2)^{-\frac{2}{4}} + (12x + 8)^{\frac{3}{2}}$

59. $(32x^3 + 48x^2 - 162x - 243)^{-1}$

LEARNING OBJECTIVES

In this section, you will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

1.6 RATIONAL EXPRESSIONS

A pastry shop has fixed costs of \$280 per week and variable costs of \$9 per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280 + 9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280 + 9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a **rational expression**. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2 + 8x + 16}{x^2 + 11x + 28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x + 4)^2}{(x + 4)(x + 7)}$$

Then we can simplify that expression by canceling the common factor $(x + 4)$.

$$\frac{x + 4}{x + 7}$$

How To...

Given a rational expression, simplify it.

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example 1 Simplifying Rational Expressions

Simplify $\dfrac{x^2 - 9}{x^2 + 4x + 3}$.

Solution

$$\frac{(x + 3)(x - 3)}{(x + 3)(x + 1)}$$ Factor the numerator and the denominator.

$$\frac{x - 3}{x + 1}$$ Cancel common factor $(x + 3)$.

Analysis *We can cancel the common factor because any expression divided by itself is equal to* 1.

Q & A...

Can the x^2 term be cancelled in Example 1?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Try It #1

Simplify $\dfrac{x - 6}{x^2 - 36}$.

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

How To…

Given two rational expressions, multiply them.

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example 2 **Multiplying Rational Expressions**

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2 + 4x - 5}{3x + 18} \cdot \frac{2x - 1}{x + 5}$$

Solution

$$\frac{(x + 5)(x - 1)}{3(x + 6)} \cdot \frac{(2x - 1)}{(x + 5)}$$ Factor the numerator and denominator.

$$\frac{(x + 5)(x - 1)(2x - 1)}{3(x + 6)(x + 5)}$$ Multiply numerators and denominators.

$$\frac{\cancel{(x + 5)}(x - 1)(2x - 1)}{3(x + 6)\cancel{(x + 5)}}$$ Cancel common factors to simplify.

$$\frac{(x - 1)(2x - 1)}{3(x + 6)}$$

Try It #2

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2 + 11x + 30}{x^2 + 5x + 6} \cdot \frac{x^2 + 7x + 12}{x^2 + 8x + 16}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

How To...

Given two rational expressions, divide them.

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example 3 Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2 + x - 6}{x^2 - 1} \div \frac{x^2 - 4}{x^2 + 2x + 1}$$

Solution

$$\frac{2x^2 + x - 6}{x^2 - 1} \cdot \frac{x^2 + 2x + 1}{x^2 - 4} \qquad \text{Rewrite as multiplication.}$$

$$\frac{(2x - 3)(x + 2)}{(x + 1)(x - 1)} \cdot \frac{(x + 1)(x + 1)}{(x + 2)(x - 2)} \qquad \text{Factor the numerator and denominator.}$$

$$\frac{(2x - 3)(x + 2)(x + 1)(x + 1)}{(x + 1)(x - 1)(x + 2)(x - 2)} \qquad \text{Multiply numerators and denominators.}$$

$$\frac{(2x - 3)\cancel{(x + 2)}\cancel{(x + 1)}(x + 1)}{\cancel{(x + 1)}(x - 1)\cancel{(x + 2)}(x - 2)} \qquad \text{Cancel common factors to simplify.}$$

$$\frac{(2x - 3)(x + 1)}{(x - 1)(x - 2)}$$

Try It #3

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2 - 16}{3x^2 + 17x - 28} \div \frac{3x^2 - 2x - 8}{x^2 + 5x - 14}$$

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\frac{5}{24} + \frac{1}{40} = \frac{25}{120} + \frac{3}{120}$$
$$= \frac{28}{120}$$
$$= \frac{7}{30}$$

We have to rewrite the fractions so they share a common denominator before we are able to add.We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x + 3)(x + 4)$ and $(x + 4)(x + 5)$, then the LCD would be $(x + 3)(x + 4)(x + 5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x + 3)(x + 4)$ by $\frac{x + 5}{x + 5}$ and the expression with a denominator of $(x + 3)(x + 4)$ by $\frac{x + 3}{x + 3}$.

How To…

Given two rational expressions, add or subtract them.

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example 4 **Adding Rational Expressions**

Add the rational expressions:

$$\frac{5}{x} + \frac{6}{y}$$

Solution First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\frac{5}{x} \cdot \frac{y}{y} + \frac{6}{y} \cdot \frac{x}{x}$$

$$\frac{5y}{xy} + \frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x + 5y}{xy}$$

Analysis *Multiplying by* $\frac{y}{y}$ *or* $\frac{x}{x}$ *does not change the value of the original expression because any number divided by itself is* 1, *and multiplying an expression by* 1 *gives the original expression.*

Example 5 **Subtracting Rational Expressions**

Subtract the rational expressions:

$$\frac{6}{x^2 + 4x + 4} - \frac{2}{x^2 - 4}$$

Solution

$\frac{6}{(x+2)^2} - \frac{2}{(x+2)(x-2)}$	Factor.
$\frac{6}{(x+2)^2} \cdot \frac{x-2}{x-2} - \frac{2}{(x+2)(x-2)} \cdot \frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator.
$\frac{6(x-2)}{(x+2)^2(x-2)} - \frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply.
$\frac{6x - 12 - (2x + 4)}{(x+2)^2(x-2)}$	Apply distributive property.
$\frac{4x - 16}{(x+2)^2(x-2)}$	Subtract.
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify.

Q & A…

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Try It #4

Subtract the rational expressions: $\dfrac{3}{x+5} - \dfrac{1}{x-3}$.

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\dfrac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\dfrac{a}{1}$ and combining the expressions in the denominator as $\dfrac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\dfrac{a}{1} \cdot \dfrac{b}{1+bc}$, which is equal to $\dfrac{ab}{1+bc}$.

How To...

Given a complex rational expression, simplify it.

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

Example 6 Simplifying Complex Rational Expressions

Simplify: $\dfrac{y+\frac{1}{x}}{\frac{x}{y}}$.

Solution Begin by combining the expressions in the numerator into one expression.

$y \cdot \dfrac{x}{x} + \dfrac{1}{x}$ Multiply by $\dfrac{x}{x}$ to get LCD as denominator.

$\dfrac{xy}{x} + \dfrac{1}{x}$

$\dfrac{xy+1}{x}$ Add numerators.

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\dfrac{\frac{xy+1}{x}}{\frac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$\dfrac{xy+1}{x} \div \dfrac{x}{y}$

$\dfrac{xy+1}{x} \cdot \dfrac{y}{x}$ Rewrite as multiplication.

$\dfrac{y(xy+1)}{x^2}$ Multiply.

Try It #5

Simplify: $\dfrac{\dfrac{x}{y} - \dfrac{y}{x}}{y}$

Q & A...

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Access these online resources for additional instruction and practice with rational expressions.

- Simplify Rational Expressions (http://openstaxcollege.org/l/simpratexpress)
- Multiply and Divide Rational Expressions (http://openstaxcollege.org/l/multdivratex)
- Add and Subtract Rational Expressions (http://openstaxcollege.org/l/addsubratex)
- Simplify a Complex Fraction (http://openstaxcollege.org/l/complexfract)

1.6 SECTION EXERCISES

VERBAL

1. How can you use factoring to simplify rational expressions?

2. How do you use the LCD to combine two rational expressions?

3. Tell whether the following statement is true or false and explain why: You only need to find the LCD when adding or subtracting rational expressions.

ALGEBRAIC

For the following exercises, simplify the rational expressions.

4. $\dfrac{x^2-16}{x^2-5x+4}$

5. $\dfrac{y^2+10y+25}{y^2+11y+30}$

6. $\dfrac{6a^2-24a+24}{6a^2-24}$

7. $\dfrac{9b^2+18b+9}{3b+3}$

8. $\dfrac{m-12}{m^2-144}$

9. $\dfrac{2x^2+7x-4}{4x^2+2x-2}$

10. $\dfrac{6x^2+5x-4}{3x^2+19x+20}$

11. $\dfrac{a^2+9a+18}{a^2+3a-18}$

12. $\dfrac{3c^2+25c-18}{3c^2-23c+14}$

13. $\dfrac{12n^2-29n-8}{28n^2-5n-3}$

For the following exercises, multiply the rational expressions and express the product in simplest form.

14. $\dfrac{x^2-x-6}{2x^2+x-6}\cdot\dfrac{2x^2+7x-15}{x^2-9}$

15. $\dfrac{c^2+2c-24}{c^2+12c+36}\cdot\dfrac{c^2-10c+24}{c^2-8c+16}$

16. $\dfrac{2d^2+9d-35}{d^2+10d+21}\cdot\dfrac{3d^2+2d-21}{3d^2+14d-49}$

17. $\dfrac{10h^2-9h-9}{2h^2-19h+24}\cdot\dfrac{h^2-16h+64}{5h^2-37h-24}$

18. $\dfrac{6b^2+13b+6}{4b^2-9}\cdot\dfrac{6b^2+31b-30}{18b^2-3b-10}$

19. $\dfrac{2d^2+15d+25}{4d^2-25}\cdot\dfrac{2d^2-15d+25}{25d^2-1}$

20. $\dfrac{6x^2-5x-50}{15x^2-44x-20}\cdot\dfrac{20x^2-7x-6}{2x^2+9x+10}$

21. $\dfrac{t^2-1}{t^2+4t+3}\cdot\dfrac{t^2+2t-15}{t^2-4t+3}$

22. $\dfrac{2n^2-n-15}{6n^2+13n-5}\cdot\dfrac{12n^2-13n+3}{4n^2-15n+9}$

23. $\dfrac{36x^2-25}{6x^2+65x+50}\cdot\dfrac{3x^2+32x+20}{18x^2+27x+10}$

For the following exercises, divide the rational expressions.

24. $\dfrac{3y^2-7y-6}{2y^2-3y-9}\div\dfrac{y^2+y-2}{2y^2+y-3}$

25. $\dfrac{6p^2+p-12}{8p^2+18p+9}\div\dfrac{6p^2-11p+4}{2p^2+11p-6}$

26. $\dfrac{q^2-9}{q^2+6q+9}\div\dfrac{q^2-2q-3}{q^2+2q-3}$

27. $\dfrac{18d^2+77d-18}{27d^2-15d+2}\div\dfrac{3d^2+29d-44}{9d^2-15d+4}$

28. $\dfrac{16x^2+18x-55}{32x^2-36x-11}\div\dfrac{2x^2+17x+30}{4x^2+25x+6}$

29. $\dfrac{144b^2-25}{72b^2-6b-10}\div\dfrac{18b^2-21b+5}{36b^2-18b-10}$

30. $\dfrac{16a^2-24a+9}{4a^2+17a-15}\div\dfrac{16a^2-9}{4a^2+11a+6}$

31. $\dfrac{22y^2+59y+10}{12y^2+28y-5}\div\dfrac{11y^2+46y+8}{24y^2-10y+1}$

32. $\dfrac{9x^2+3x-20}{3x^2-7x+4}\div\dfrac{6x^2+4x-10}{x^2-2x+1}$

For the following exercises, add and subtract the rational expressions, and then simplify.

33. $\dfrac{4}{x} + \dfrac{10}{y}$

34. $\dfrac{12}{2q} - \dfrac{6}{3p}$

35. $\dfrac{4}{a+1} + \dfrac{5}{a-3}$

36. $\dfrac{c+2}{3} - \dfrac{c-4}{4}$

37. $\dfrac{y+3}{y-2} + \dfrac{y-3}{y+1}$

38. $\dfrac{x-1}{x+1} - \dfrac{2x+3}{2x+1}$

39. $\dfrac{3z}{z+1} + \dfrac{2z+5}{z-2}$

40. $\dfrac{4p}{p+1} - \dfrac{p+1}{4p}$

41. $\dfrac{x}{x+1} + \dfrac{y}{y+1}$

For the following exercises, simplify the rational expression.

42. $\dfrac{\frac{6}{y} - \frac{4}{x}}{y}$

43. $\dfrac{\frac{2}{a} + \frac{7}{b}}{b}$

44. $\dfrac{\frac{x}{4} - \frac{p}{8}}{p}$

45. $\dfrac{\frac{3}{a} + \frac{b}{6}}{\frac{2b}{3a}}$

46. $\dfrac{\frac{3}{x+1} + \frac{2}{x-1}}{\frac{x-1}{x+1}}$

47. $\dfrac{\frac{a}{b} - \frac{b}{a}}{\frac{a+b}{ab}}$

48. $\dfrac{\frac{2x}{3} + \frac{4x}{7}}{\frac{x}{2}}$

49. $\dfrac{\frac{2c}{c+2} + \frac{c-1}{c+1}}{\frac{2c+1}{c+1}}$

50. $\dfrac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x}}$

REAL-WORLD APPLICATIONS

51. Brenda is placing tile on her bathroom floor. The area of the floor is $15x^2 - 8x - 7$ ft². The area of one tile is $x^2 - 2x + 1$ ft². To find the number of tiles needed, simplify the rational expression: $\dfrac{15x^2 - 8x - 7}{x^2 - 2x + 1}$.

Area $= 15x^2 - 8x - 7$

52. The area of Sandy's yard is $25x^2 - 625$ ft². A patch of sod has an area of $x^2 - 10x + 25$ ft². Divide the two areas and simplify to find how many pieces of sod Sandy needs to cover her yard.

53. Aaron wants to mulch his garden. His garden is $x^2 + 18x + 81$ ft². One bag of mulch covers $x^2 - 81$ ft². Divide the expressions and simplify to find how many bags of mulch Aaron needs to mulch his garden.

EXTENSIONS

For the following exercises, perform the given operations and simplify.

54. $\dfrac{x^2 + x - 6}{x^2 - 2x - 3} \cdot \dfrac{2x^2 - 3x - 9}{x^2 - x - 2} \div \dfrac{10x^2 + 27x + 18}{x^2 + 2x + 1}$

55. $\dfrac{\frac{3y^2 - 10y + 3}{3y^2 + 5y - 2} \cdot \frac{2y^2 - 3y - 20}{2y^2 - y - 15}}{y - 4}$

56. $\dfrac{\frac{4a+1}{2a-3} + \frac{2a-3}{2a+3}}{\frac{4a^2+9}{a}}$

57. $\dfrac{x^2 + 7x + 12}{x^2 + x - 6} \div \dfrac{3x^2 + 19x + 28}{8x^2 - 4x - 24} \div \dfrac{2x^2 + x - 3}{3x^2 + 4x - 7}$

CHAPTER 1 REVIEW

Key Terms

algebraic expression constants and variables combined using addition, subtraction, multiplication, and division

associative property of addition the sum of three numbers may be grouped differently without affecting the result; in symbols, $a + (b + c) = (a + b) + c$

associative property of multiplication the product of three numbers may be grouped differently without affecting the result; in symbols, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

base in exponential notation, the expression that is being multiplied

binomial a polynomial containing two terms

coefficient any real number a_i in a polynomial in the form $a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

commutative property of addition two numbers may be added in either order without affecting the result; in symbols, $a + b = b + a$

commutative property of multiplication two numbers may be multiplied in any order without affecting the result; in symbols, $a \cdot b = b \cdot a$

constant a quantity that does not change value

degree the highest power of the variable that occurs in a polynomial

difference of squares the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign

distributive property the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a \cdot (b + c) = a \cdot b + a \cdot c$

equation a mathematical statement indicating that two expressions are equal

exponent in exponential notation, the raised number or variable that indicates how many times the base is being multiplied

exponential notation a shorthand method of writing products of the same factor

factor by grouping a method for factoring a trinomial in the form $ax^2 + bx + c$ by dividing the x term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression

formula an equation expressing a relationship between constant and variable quantities

greatest common factor the largest polynomial that divides evenly into each polynomial

identity property of addition there is a unique number, called the additive identity, 0, which, when added to a number, results in the original number; in symbols, $a + 0 = a$

identity property of multiplication there is a unique number, called the multiplicative identity, 1, which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1 = a$

index the number above the radical sign indicating the nth root

integers the set consisting of the natural numbers, their opposites, and 0: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

inverse property of addition for every real number a, there is a unique number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0; in symbols, $a + (-a) = 0$

inverse property of multiplication for every non-zero real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1; in symbols, $a \cdot \frac{1}{a} = 1$

irrational numbers the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers

leading coefficient the coefficient of the leading term

leading term the term containing the highest degree

least common denominator the smallest multiple that two denominators have in common

monomial a polynomial containing one term

natural numbers the set of counting numbers: $\{1, 2, 3, \ldots\}$

order of operations a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations

perfect square trinomial the trinomial that results when a binomial is squared

polynomial a sum of terms each consisting of a variable raised to a nonnegative integer power

principal *n*th root the number with the same sign as a that when raised to the nth power equals a

principal square root the nonnegative square root of a number a that, when multiplied by itself, equals a

radical the symbol used to indicate a root

radical expression an expression containing a radical symbol

radicand the number under the radical symbol

rational expression the quotient of two polynomial expressions

rational numbers the set of all numbers of the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.

real number line a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0; positive numbers lie to the right of 0 and negative numbers to the left.

real numbers the sets of rational numbers and irrational numbers taken together

scientific notation a shorthand notation for writing very large or very small numbers in the form $a \times 10^n$ where $1 \leq |a| < 10$ and n is an integer

term of a polynomial any $a_i x^i$ of a polynomial in the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$

trinomial a polynomial containing three terms

variable a quantity that may change value

whole numbers the set consisting of 0 plus the natural numbers: $\{0, 1, 2, 3, \ldots\}$

Key Equations

Rules of Exponents

For nonzero real numbers a and b and integers m and n

Product rule	$a^m \cdot a^n = a^{m+n}$
Quotient rule	$\dfrac{a^m}{a^n} = a^{m-n}$
Power rule	$(a^m)^n = a^{m \cdot n}$
Zero exponent rule	$a^0 = 1$
Negative rule	$a^{-n} = \dfrac{1}{a^n}$
Power of a product rule	$(a \cdot b)^n = a^n \cdot b^n$
Power of a quotient rule	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
perfect square trinomial	$(x + a)^2 = (x + a)(x + a) = x^2 + 2ax + a^2$
difference of squares	$(a + b)(a - b) = a^2 - b^2$
difference of squares	$a^2 - b^2 = (a + b)(a - b)$
perfect square trinomial	$a^2 + 2ab + b^2 = (a + b)^2$
sum of cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
difference of cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Key Concepts

1.1 Real Numbers: Algebra Essentials

- Rational numbers may be written as fractions or terminating or repeating decimals. See **Example 1** and **Example 2**.
- Determine whether a number is rational or irrational by writing it as a decimal. See **Example 3**.
- The rational numbers and irrational numbers make up the set of real numbers. See **Example 4**. A number can be classified as natural, whole, integer, rational, or irrational. See **Example 5**.
- The order of operations is used to evaluate expressions. See **Example 6**.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties. See **Example 7**.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. See **Example 8**. They take on a numerical value when evaluated by replacing variables with constants. See **Example 9**, **Example 10**, and **Example 12.**
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression. See **Example 11** and **Example 13**.

1.2 Exponents and Scientific Notation

- Products of exponential expressions with the same base can be simplified by adding exponents. See **Example 1**.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents. See **Example 2**.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents. See **Example 3**.
- An expression with exponent zero is defined as 1. See **Example 4**.
- An expression with a negative exponent is defined as a reciprocal. See **Example 5** and **Example 6**.
- The power of a product of factors is the same as the product of the powers of the same factors. See **Example 7**.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors. See **Example 8**.
- The rules for exponential expressions can be combined to simplify more complicated expressions. See **Example 9**.
- Scientific notation uses powers of 10 to simplify very large or very small numbers. See **Example 10** and **Example 11**.
- Scientific notation may be used to simplify calculations with very large or very small numbers. See **Example 12** and **Example 13**.

1.3 Radicals and Rational Expressions

- The principal square root of a number a is the nonnegative number that when multiplied by itself equals a. See **Example 1**.
- If a and b are nonnegative, the square root of the product ab is equal to the product of the square roots of a and b See **Example 2** and **Example 3**.
- If a and b are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of a and b See **Example 4** and **Example 5**.
- We can add and subtract radical expressions if they have the same radicand and the same index. See **Example 6** and **Example 7**.
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator. See **Example 8** and **Example 9**.
- The principal nth root of a is the number with the same sign as a that when raised to the nth power equals a. These roots have the same properties as square roots. See **Example 10**.
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals. See **Example 11** and **Example 12**.
- The properties of exponents apply to rational exponents. See **Example 13**.

1.4 Polynomials

- A polynomial is a sum of terms each consisting of a variable raised to a non-negative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term. See **Example 1**.
- We can add and subtract polynomials by combining like terms. See **Example 2** and **Example 3.**
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products. See **Example 4**.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials. See **Example 5**.
- Perfect square trinomials and difference of squares are special products. See **Example 6** and **Example 7**.
- Follow the same rules to work with polynomials containing several variables. See **Example 8**.

1.5 Factoring Polynomials

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem. See **Example 1**.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term. See **Example 2**.
- Trinomials can be factored using a process called factoring by grouping. See **Example 3**.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations. See **Example 4** and **Example 5**.
- The sum of cubes and the difference of cubes can be factored using equations. See **Example 6** and **Example 7**.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF. See **Example 8**.

1.6 Rational Expressions

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See **Example 1**.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See **Example 2**.
- To divide rational expressions, multiply by the reciprocal of the second expression. See **Example 3**.
- Adding or subtracting rational expressions requires finding a common denominator. See **Example 4** and **Example 5**.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See **Example 6**.

CHAPTER 1 REVIEW EXERCISES

REAL NUMBERS: ALGEBRA ESSENTIALS

For the following exercises, perform the given operations.

1. $(5 - 3 \cdot 2)^2 - 6$

2. $64 \div (2 \cdot 8) + 14 \div 7$

3. $2 \cdot 5^2 + 6 \div 2$

For the following exercises, solve the equation.

4. $5x + 9 = -11$

5. $2y + 4^2 = 64$

For the following exercises, simplify the expression.

6. $9(y + 2) \div 3 \cdot 2 + 1$

7. $3m(4 + 7) - m$

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

8. 11

9. 0

10. $\frac{5}{6}$

11. $\sqrt{11}$

EXPONENTS AND SCIENTIFIC NOTATION

For the following exercises, simplify the expression.

12. $2^2 \cdot 2^4$

13. $\frac{4^5}{4^3}$

14. $\left(\frac{a^2}{b^3}\right)^4$

15. $\frac{6a^2 \cdot a^0}{2a^{-4}}$

16. $\frac{(xy)^4}{y^3} \cdot \frac{2}{x^5}$

17. $\frac{4^{-2}x^3y^{-3}}{2x^0}$

18. $\left(\frac{2x^2}{y}\right)^{-2}$

19. $\left(\frac{16a^3}{b^2}\right)(4ab^{-1})^{-2}$

20. Write the number in standard notation: 2.1314×10^{-6}

21. Write the number in scientific notation: 16,340,000

RADICALS AND RATIONAL EXPRESSIONS

For the following exercises, find the principal square root.

22. $\sqrt{121}$

23. $\sqrt{196}$

24. $\sqrt{361}$

25. $\sqrt{75}$

26. $\sqrt{162}$

27. $\sqrt{\frac{32}{25}}$

28. $\sqrt{\frac{80}{81}}$

29. $\sqrt{\frac{49}{1250}}$

30. $\frac{2}{4 + \sqrt{2}}$

31. $4\sqrt{3} + 6\sqrt{3}$

32. $12\sqrt{5} - 13\sqrt{5}$

33. $\sqrt[5]{-243}$

34. $\frac{\sqrt[3]{250}}{\sqrt[3]{-8}}$

POLYNOMIALS

For the following exercises, perform the given operations and simplify.

35. $(3x^3 + 2x - 1) + (4x^2 - 2x + 7)$

36. $(2y + 1) - (2y^2 - 2y - 5)$

37. $(2x^2 + 3x - 6) + (3x^2 - 4x + 9)$

38. $(6a^2 + 3a + 10) - (6a^2 - 3a + 5)$

39. $(k + 3)(k - 6)$

40. $(2h + 1)(3h - 2)$

41. $(x + 1)(x^2 + 1)$

42. $(m - 2)(m^2 + 2m - 3)$

43. $(a + 2b)(3a - b)$

44. $(x + y)(x - y)$

FACTORING POLYNOMIALS

For the following exercises, find the greatest common factor.

45. $81p + 9pq - 27p^2q^2$

46. $12x^2y + 4xy^2 - 18xy$

47. $88a^3b + 4a^2b - 144a^2$

For the following exercises, factor the polynomial.

48. $2x^2 - 9x - 18$

49. $8a^2 + 30a - 27$

50. $d^2 - 5d - 66$

51. $x^2 + 10x + 25$

52. $y^2 - 6y + 9$

53. $4h^2 - 12hk + 9k^2$

54. $361x^2 - 121$

55. $p^3 + 216$

56. $8x^3 - 125$

57. $64q^3 - 27p^3$

58. $4x(x - 1)^{-\frac{1}{4}} + 3(x - 1)^{\frac{3}{4}}$

59. $3p(p + 3)^{\frac{1}{3}} - 8(p + 3)^{\frac{4}{3}}$

60. $4r(2r - 1)^{-\frac{2}{3}} - 5(2r - 1)^{\frac{1}{3}}$

RATIONAL EXPRESSIONS

For the following exercises, simplify the expression.

61. $\dfrac{x^2 - x - 12}{x^2 - 8x + 16}$

62. $\dfrac{4y^2 - 25}{4y^2 - 20y + 25}$

63. $\dfrac{2a^2 - a - 3}{2a^2 - 6a - 8} \cdot \dfrac{5a^2 - 19a - 4}{10a^2 - 13a - 3}$

64. $\dfrac{d - 4}{d^2 - 9} \cdot \dfrac{d - 3}{d^2 - 16}$

65. $\dfrac{m^2 + 5m + 6}{2m^2 - 5m - 3} \div \dfrac{2m^2 + 3m - 9}{4m^2 - 4m - 3}$

66. $\dfrac{4d^2 - 7d - 2}{6d^2 - 17d + 10} \div \dfrac{8d^2 + 6d + 1}{6d^2 + 7d - 10}$

67. $\dfrac{10}{x} + \dfrac{6}{y}$

68. $\dfrac{12}{a^2 + 2a + 1} - \dfrac{3}{a^2 - 1}$

69. $\dfrac{\frac{1}{d} + \frac{2}{c}}{\frac{6c + 12d}{dc}}$

70. $\dfrac{\frac{3}{x} - \frac{7}{y}}{\frac{2}{x}}$

CHAPTER 1 PRACTICE TEST

For the following exercises, identify the number as rational, irrational, whole, or natural. Choose the most descriptive answer.

1. -13

2. $\sqrt{2}$

For the following exercises, evaluate the equations.

3. $2(x+3)-12=18$

4. $y(3+3)^2-26=10$

5. Write the number in standard notation: 3.1415×10^6

6. Write the number in scientific notation: 0.0000000212.

For the following exercises, simplify the expression.

7. $-2 \cdot (2+3 \cdot 2)^2 + 144$

8. $4(x+3)-(6x+2)$

9. $3^5 \cdot 3^{-3}$

10. $\left(\frac{2}{3}\right)^3$

11. $\frac{8x^3}{(2x)^2}$

12. $(16y^0)2y^{-2}$

13. $\sqrt{441}$

14. $\sqrt{490}$

15. $\sqrt{\frac{9x}{16}}$

16. $\frac{\sqrt{121b^2}}{1+\sqrt{b}}$

17. $6\sqrt{24}+7\sqrt{54}-12\sqrt{6}$

18. $\frac{\sqrt[3]{-8}}{\sqrt[4]{625}}$

19. $(13q^3+2q^2-3)-(6q^2+5q-3)$

20. $(6p^2+2p+1)+(9p^2-1)$

21. $(n-2)(n^2-4n+4)$

22. $(a-2b)(2a+b)$

For the following exercises, factor the polynomial.

23. $16x^2-81$

24. $y^2+12y+36$

25. $27c^3-1331$

26. $3x(x-6)^{-\frac{1}{4}}+2(x-6)^{\frac{3}{4}}$

For the following exercises, simplify the expression.

27. $\frac{2z^2+7z+3}{z^2-9} \cdot \frac{4z^2-15z+9}{4z^2-1}$

28. $\frac{x}{y}+\frac{2}{x}$

29. $\frac{\frac{a}{2b}-\frac{2b}{9a}}{\frac{3a-2b}{6a}}$