

5

Polynomial and Rational Functions

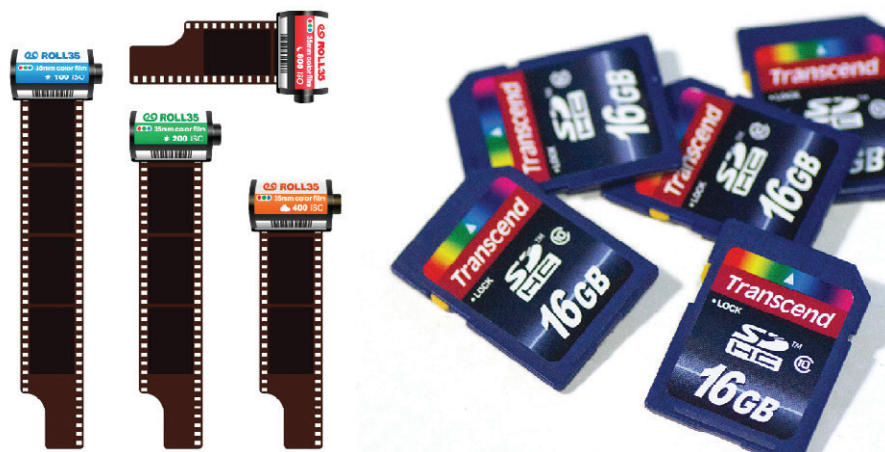


Figure 1 35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit “film”: modification of work by Horia Varlan; credit “memory cards”: modification of work by Paul Hudson)

CHAPTER OUTLINE

- 5.1 Quadratic Functions
- 5.2 Power Functions and Polynomial Functions
- 5.3 Graphs of Polynomial Functions
- 5.4 Dividing Polynomials
- 5.5 Zeros of Polynomial Functions
- 5.6 Rational Functions
- 5.7 Inverses and Radical Functions
- 5.8 Modeling Using Variation

Introduction

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

LEARNING OBJECTIVES

In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

5.1 QUADRATIC FUNCTIONS

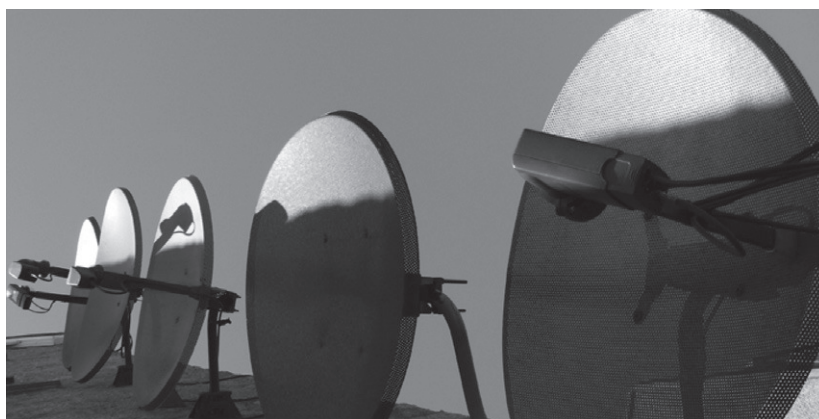


Figure 1 An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in **Figure 1** are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in **Figure 2**.

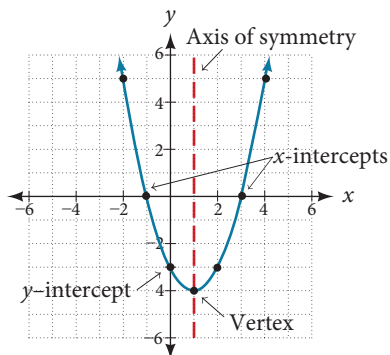


Figure 2

The y -intercept is the point at which the parabola crosses the y -axis. The x -intercepts are the points at which the parabola crosses the x -axis. If they exist, the x -intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 1 Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y -intercept of the parabola shown in **Figure 3**.

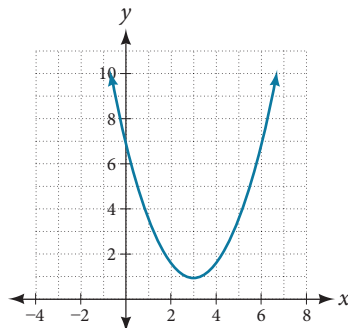


Figure 3

Solution The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x -axis, so it has no zeros. It crosses the y -axis at $(0, 7)$ so this is the y -intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c$$

where a , b , and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x -intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x -intercepts, those points where the parabola crosses the x -axis, occur at $(-3, 0)$ and $(-1, 0)$.

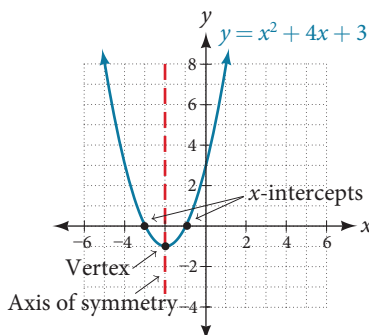


Figure 4

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. **Figure 5** represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

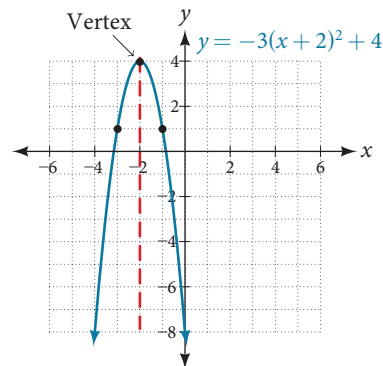


Figure 5

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. **Figure 6** is the graph of this basic function.

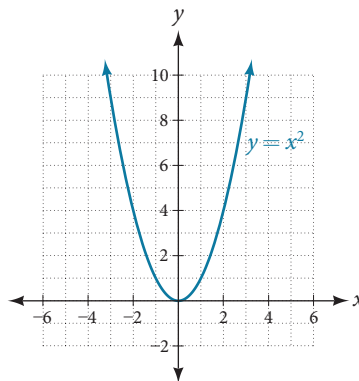


Figure 6

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In **Figure 5**, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In **Figure 5**, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x -value shifts farther from the x -axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x -value shifts closer to the x -axis, so the graph appears to become wider, but in fact there is a vertical compression. In **Figure 5**, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$\begin{aligned} a(x - h)^2 + k &= ax^2 + bx + c \\ ax^2 - 2ahx + (ah^2 + k) &= ax^2 + bx + c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b, \text{ so } h = -\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2 + k &= c \\ k &= c - ah^2 \\ &= c - a\left(\frac{b}{2a}\right)^2 \\ &= c - \frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h , so $f(h) = k$.

forms of quadratic functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where a , b , and c are real numbers and $a \neq 0$.

The **standard form of a quadratic function** is $f(x) = a(x - h)^2 + k$ where $a \neq 0$.

The vertex (h, k) is located at

$$h = -\frac{b}{2a}, k = f(h) = f\left(\frac{-b}{2a}\right).$$

How To...

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h . Identify the vertical shift of the parabola; this value is k .
2. Substitute the values of the horizontal and vertical shift for h and k in the function $f(x) = a(x - h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a > 0$. If the parabola opens down, $a < 0$ since this means the graph was reflected about the x -axis.
6. Expand and simplify to write in general form.

Example 2 Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in **Figure 7** as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

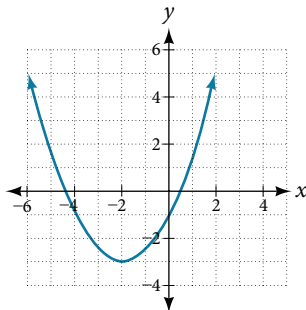


Figure 7

Solution We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x - (-2))^2 - 3 = a(x + 2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0 + 2)^2 - 3$$

$$2 = 4a$$

$$a = \frac{1}{2}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x + 2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x + 2)^2 - 3 \\ &= \frac{1}{2}(x + 2)(x + 2) - 3 \\ &= \frac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis We can check our work using the table feature on a graphing utility. First enter $Y1 = \frac{1}{2}(x + 2)^2 - 3$. Next, select **TBLSET**, then use **TblStart** = -6 and $\Delta Tbl = 2$, and select **TABLE**. See **Table 1**.

x	-6	-4	-2	0	2
y	5	-1	-3	-1	5

Table 1

The ordered pairs in the table correspond to points on the graph.

Try It #1

A coordinate grid has been superimposed over the quadratic path of a basketball in **Figure 8**. Find an equation for the path of the ball. Does the shooter make the basket?

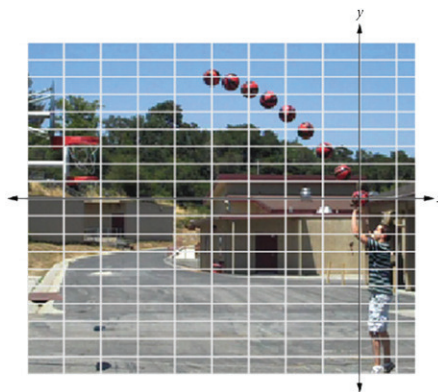


Figure 8 (credit: modification of work by Dan Meyer)

How To...

Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a , b , and c .
2. Find h , the x -coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k , the y -coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 3 Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the “ a ” from the general form.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(x) &= 2x^2 - 6x + 7 \end{aligned}$$

The standard form of a quadratic function prior to writing the function then becomes the following:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Analysis One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k , and where it occurs, x .

Try It #2

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y -values greater than or equal to the y -coordinate at the turning point or less than or equal to the y -coordinate at the turning point, depending on whether the parabola opens up or down.

domain and range of a quadratic function

The domain of any quadratic function is all real numbers unless the context of the function presents some restrictions. The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is

$$f(x) \geq f\left(-\frac{b}{2a}\right), \text{ or } \left[f\left(-\frac{b}{2a}\right), \infty\right).$$

The range of a quadratic function written in general form with a negative a value is $f(x) \leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form $f(x) = a(x - h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To...

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k .
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 4 Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x -value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $(-\infty, \frac{61}{20}]$.

Try It #3

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in **Figure 9**.

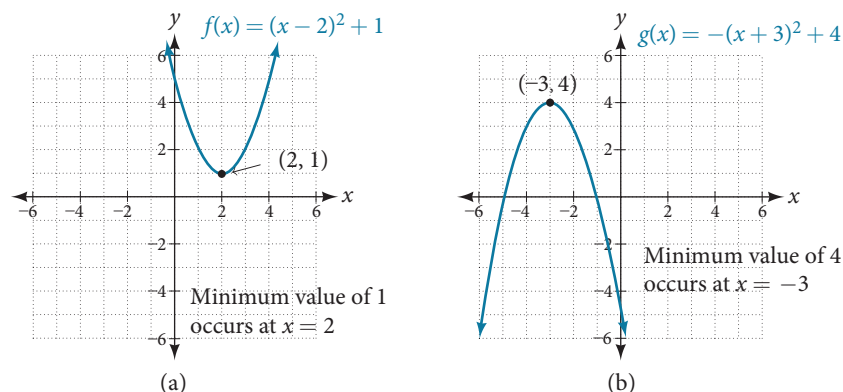


Figure 9

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 5 Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

- Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L .
- What dimensions should she make her garden to maximize the enclosed area?

Solution Let's use a diagram such as **Figure 10** to record the given information. It is also helpful to introduce a temporary variable, W , to represent the width of the garden and the length of the fence section parallel to the backyard fence.

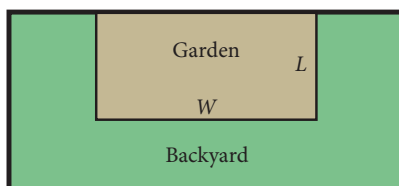


Figure 10

- We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W , in terms of L .

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$\begin{aligned} A &= LW = L(80 - 2L) \\ A(L) &= 80L - 2L^2 \end{aligned}$$

This formula represents the area of the fence in terms of the variable length L . The function, written in general form, is

$$A(L) = -2L^2 + 80L.$$

- The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{b}{2a} & k &= A(20) \\ h &= -\frac{80}{2(-2)} & &= 80(20) - 2(20)^2 \\ &= 20 & \text{and} & = 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in **Figure 11**.

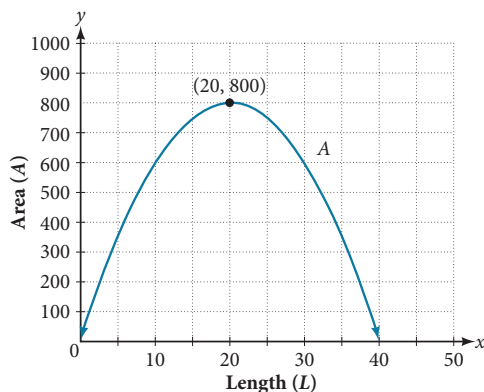


Figure 11

How To...

Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for a revenue function.
2. Find the vertex of the quadratic equation.
3. Determine the y -value of the vertex.

Example 6 Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation Revenue = pQ .

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to \$32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$\begin{aligned} m &= \frac{79,000 - 84,000}{32 - 30} \\ &= \frac{-5,000}{2} \\ &= -2,500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y -intercept.

$$\begin{aligned} Q &= -2,500p + b && \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \\ 84,000 &= -2,500(30) + b && \text{Solve for } b \\ b &= 159,000 \end{aligned}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= pQ \\ \text{Revenue} &= p(-2,500p + 159,000) \\ \text{Revenue} &= -2,500p^2 + 159,000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$\begin{aligned} h &= -\frac{159,000}{2(-2,500)} \\ &= 31.8 \end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\begin{aligned} \text{maximum revenue} &= -2,500(31.8)^2 + 159,000(31.8) \\ &= 2,528,100 \end{aligned}$$

Analysis This could also be solved by graphing the quadratic as in **Figure 12**. We can see the maximum revenue on a graph of the quadratic function.

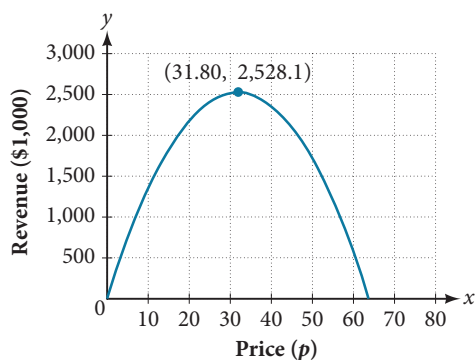
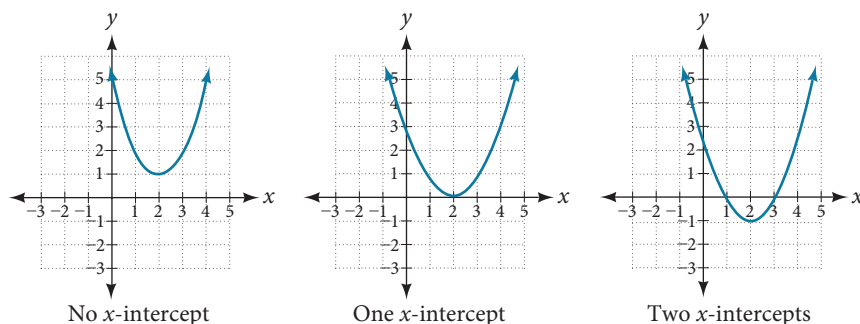


Figure 12

Finding the x - and y -Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y -intercept of a quadratic by evaluating the function at an input of zero, and we find the x -intercepts at locations where the output is zero. Notice in **Figure 13** that the number of x -intercepts can vary depending upon the location of the graph.

Figure 13 Number of x -intercepts of a parabola

How To...

Given a quadratic function $f(x)$, find the y - and x -intercepts.

1. Evaluate $f(0)$ to find the y -intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x -intercepts.

Example 7 Finding the y - and x -Intercepts of a Parabola

Find the y - and x -intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution We find the y -intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

So the y -intercept is at $(0, -2)$.

For the x -intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1 \quad 0 = x + 2$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

So the x -intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

Analysis By graphing the function, we can confirm that the graph crosses the y -axis at $(0, -2)$. We can also confirm that the graph crosses the x -axis at $(\frac{1}{3}, 0)$ and $(-2, 0)$. See **Figure 14**.

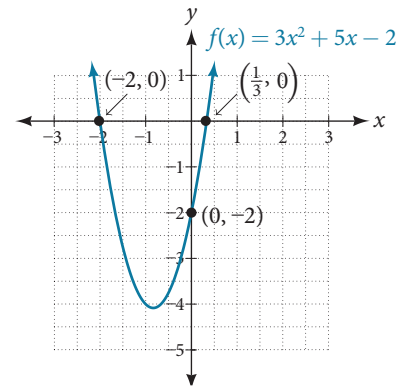


Figure 14

Rewriting Quadratics in Standard Form

In **Example 7**, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How To...

Given a quadratic function, find the x -intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k .
3. Rewrite the quadratic in standard form using h and k .
4. Solve for when the output of the function will be zero to find the x -intercepts.

Example 8 Finding the x -Intercepts of a Parabola

Find the x -intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for h and k .

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x + 1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x + 1)^2 - 6 \\ 6 &= 2(x + 1)^2 \\ 3 &= (x + 1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm\sqrt{3} \end{aligned}$$

The graph has x -intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

We can check our work by graphing the given function on a graphing utility and observing the x -intercepts. See **Figure 15**.

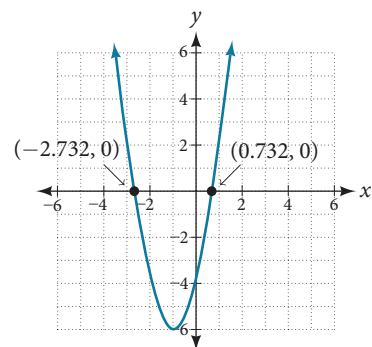


Figure 15

Analysis We could have achieved the same results using the quadratic formula. Identify $a = 2$, $b = 4$, and $c = -4$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-4)}}{2(2)} \\ x &= \frac{-4 \pm \sqrt{48}}{4} \\ x &= \frac{-4 \pm \sqrt{3(16)}}{4} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

So the x -intercepts occur at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

Try It #4

In a separate **Try It**, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y - and x -intercepts (if any).

Example 9 Applying the Vertex and x -Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

- When does the ball reach the maximum height?
- What is the maximum height of the ball?
- When does the ball hit the ground?

Solution

- The ball reaches the maximum height at the vertex of the parabola.

$$\begin{aligned} h &= -\frac{80}{2(-16)} \\ &= \frac{80}{32} \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

- b. To find the maximum height, find the y -coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H(2.5) \\ &= -16(2.5)^2 + 80(2.5) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

- c. To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$. We use the quadratic formula.

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See **Figure 16**.

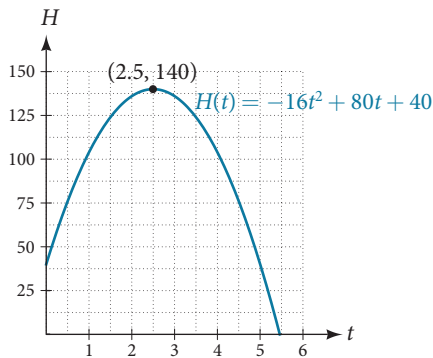


Figure 16

Notice that the graph does not represent the physical path of the ball upward and downward. Keep quantities on each axis in mind while interpreting the graph.

Try It #5

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

- When does the rock reach the maximum height?
- What is the maximum height of the rock?
- When does the rock hit the ocean?

Access these online resources for additional instruction and practice with quadratic equations.

- [Graphing Quadratic Functions in General Form \(http://openstaxcollege.org/l/graphquadgen\)](http://openstaxcollege.org/l/graphquadgen)
- [Graphing Quadratic Functions in Standard Form \(http://openstaxcollege.org/l/graphquadstan\)](http://openstaxcollege.org/l/graphquadstan)
- [Quadratic Function Review \(http://openstaxcollege.org/l/quadfuncrev\)](http://openstaxcollege.org/l/quadfuncrev)
- [Characteristics of a Quadratic Function \(http://openstaxcollege.org/l/characterquad\)](http://openstaxcollege.org/l/characterquad)

5.1 SECTION EXERCISES

VERBAL

1. Explain the advantage of writing a quadratic function in standard form.
2. How can the vertex of a parabola be used in solving real-world problems?
3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.
4. What is another name for the standard form of a quadratic function?
5. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

ALGEBRAIC

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

- | | | |
|----------------------------|-----------------------------|----------------------------|
| 6. $f(x) = x^2 - 12x + 32$ | 7. $g(x) = x^2 + 2x - 3$ | 8. $f(x) = x^2 - x$ |
| 9. $f(x) = x^2 + 5x - 2$ | 10. $h(x) = 2x^2 + 8x - 10$ | 11. $k(x) = 3x^2 - 6x - 9$ |
| 12. $f(x) = 2x^2 - 6x$ | 13. $f(x) = 3x^2 - 5x - 1$ | |

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

- | | | |
|---------------------------------------|-----------------------------|--------------------------------------|
| 14. $y(x) = 2x^2 + 10x + 12$ | 15. $f(x) = 2x^2 - 10x + 4$ | 16. $f(x) = -x^2 + 4x + 3$ |
| 17. $f(x) = 4x^2 + x - 1$ | 18. $h(t) = -4t^2 + 6t - 1$ | 19. $f(x) = \frac{1}{2}x^2 + 3x + 1$ |
| 20. $f(x) = -\frac{1}{3}x^2 - 2x + 3$ | | |

For the following exercises, determine the domain and range of the quadratic function.

- | | | |
|----------------------------|------------------------------|---------------------------|
| 21. $f(x) = (x - 3)^2 + 2$ | 22. $f(x) = -2(x + 3)^2 - 6$ | 23. $f(x) = x^2 + 6x + 4$ |
| 24. $f(x) = 2x^2 - 4x + 2$ | 25. $k(x) = 3x^2 - 6x - 9$ | |

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

- | | | |
|---|---|---|
| 26. $(h, k) = (2, 0), (x, y) = (4, 4)$ | 27. $(h, k) = (-2, -1), (x, y) = (-4, 3)$ | 28. $(h, k) = (0, 1), (x, y) = (2, 5)$ |
| 29. $(h, k) = (2, 3), (x, y) = (5, 12)$ | 30. $(h, k) = (-5, 3), (x, y) = (2, 9)$ | 31. $(h, k) = (3, 2), (x, y) = (10, 1)$ |
| 32. $(h, k) = (0, 1), (x, y) = (1, 0)$ | 33. $(h, k) = (1, 0), (x, y) = (0, 1)$ | |

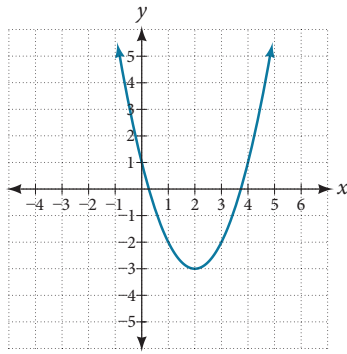
GRAPHICAL

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

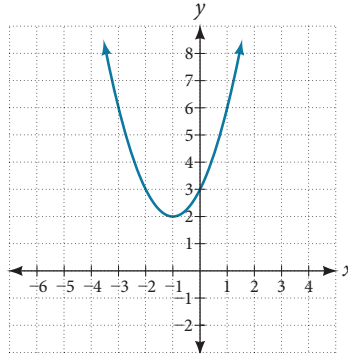
- | | | |
|---------------------------|-----------------------------|-----------------------------|
| 34. $f(x) = x^2 - 2x$ | 35. $f(x) = x^2 - 6x - 1$ | 36. $f(x) = x^2 - 5x - 6$ |
| 37. $f(x) = x^2 - 7x + 3$ | 38. $f(x) = -2x^2 + 5x - 8$ | 39. $f(x) = 4x^2 - 12x - 3$ |

For the following exercises, write the equation for the graphed function.

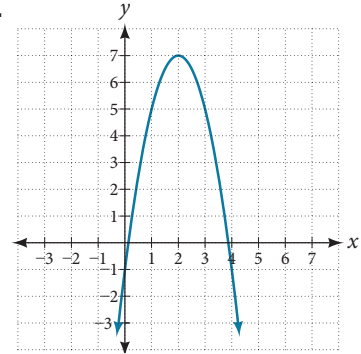
40.



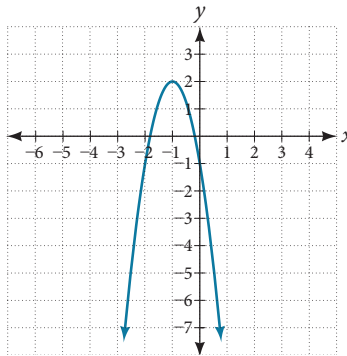
41.



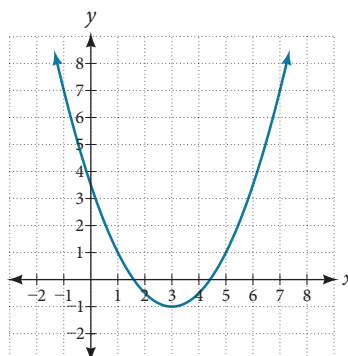
42.



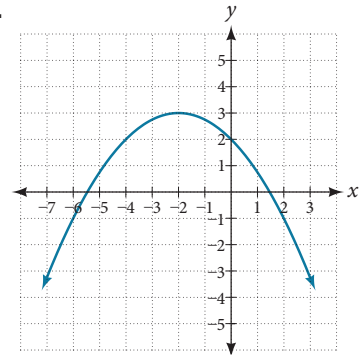
43.



44.



45.



NUMERIC

For the following exercises, use the table of values that represent points on the graph of a quadratic function.

By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

46.

x	-2	-1	0	1	2
y	5	2	1	2	5

47.

x	-2	-1	0	1	2
y	1	0	1	4	9

48.

x	-2	-1	0	1	2
y	-2	1	2	1	-2

49.

x	-2	-1	0	1	2
y	-8	-3	0	1	0

50.

x	-2	-1	0	1	2
y	8	2	0	2	8

TECHNOLOGY

For the following exercises, use a calculator to find the answer.

51. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

53. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x - 2)^2$, $f(x) = (x - 3)^2$, and $f(x) = (x + 4)^2$. What appears to be the effect of adding or subtracting those numbers?

52. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$ and $f(x) = x^2$, $f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

54. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = \frac{-32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use the [TRACE] feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

55. A suspension bridge can be modeled by the quadratic function $h(x) = 0.0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the [TRACE] feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

EXTENSIONS

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

56. Vertex $(1, -2)$, opens up.
 57. Vertex $(-1, 2)$ opens down.
 58. Vertex $(-5, 11)$, opens down.
 59. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

60. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$.
 Vertex is on the y -axis.
 61. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$.
 Vertex is on the y -axis.
 62. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$.
 Vertex is on the y -axis.
 63. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$.
 Vertex is on the y -axis.
 64. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$.
 Vertex is on the y -axis.
 65. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$.
 Vertex has x -coordinate of -1 .

REAL-WORLD APPLICATIONS

66. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.
 67. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.
 68. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.
 69. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?
 70. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?
 71. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.
 72. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.
 73. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?
 74. A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?
 75. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?