6

Exponential and Logarithmic Functions

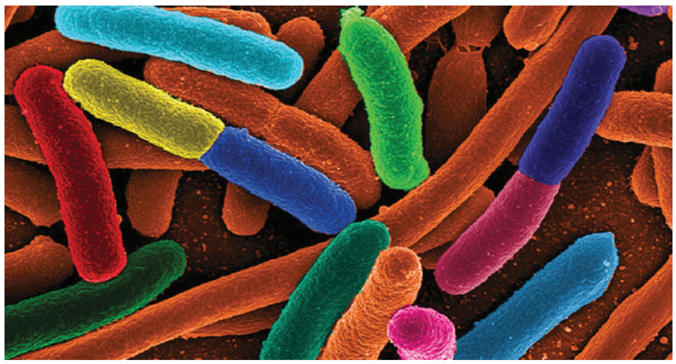

Figure 1 Electron micrograph of *E. Coli* bacteria (credit: "Mattosaurus," Wikimedia Commons)

CHAPTER OUTLINE

Introduction

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.[16]

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. **Table 1** shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

Table 1

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

16 Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.

LEARNING OBJECTIVES

In this section, you will:

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

6.5 LOGARITHMIC PROPERTIES

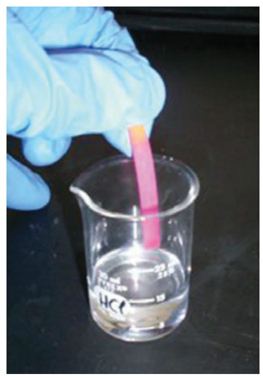

Figure 1 The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan)

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at 25° C)
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where a is the concentration of hydrogen ion in the solution

$$\begin{aligned}\text{pH} &= -\log([H^+])\\ &= \log\left(\frac{1}{([H^+]}\right)\end{aligned}$$

The equivalence of $-\log([H^+])$ and $\log\left(\frac{1}{[H^+]}\right)$ is one of the logarithm properties we will examine in this section.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b(1) = 0$$

$$\log_b(b) = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$
$$b^{\log_b(x)} = x, x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e(7)}$, and then apply the inverse property $b^{\log_b(x)} = x$ to get $e^{\log_e(7)} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

$3x = 2x + 5$	Set the arguments equal.
$x = 5$	Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b(M)$ and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$\log_b(MN) = \log_b(b^m b^n)$	Substitute for M and N.
$= \log_b(b^{m+n})$	Apply the product rule for exponents.
$= m + n$	Apply the inverse property of logs.
$= \log_b(M) + \log_b(N)$	Substitute for m and n.

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b(w) + \log_b(x) + \log_b(y) + \log_b(z)$$

the product rule for logarithms

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

How To...

Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms.

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 1 **Using the Product Rule for Logarithms**

Expand $\log_3(30x(3x+4))$.

Solution We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

Try It #1

Expand $\log_b(8k)$.

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b(M)$ and $n = \log_b(N)$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b(M) - \log_b(N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned} \log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors.} \end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned} \log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3) \end{aligned}$$

the quotient rule for logarithms

The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

How To...

Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms.

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 2 Using the Quotient Rule for Logarithms

Expand $\log_2\left(\dfrac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] - [\log_2(3x+4) + \log_2(2-x)]\\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) - \log_2(3x+4) - \log_2(2-x)\end{aligned}$$

Analysis *There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.*

Try It #2

Expand $\log_3\left(\dfrac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x)\\ &= \log_b(x) + \log_b(x)\\ &= 2\log_b(x)\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \qquad \sqrt{3} = 3^{\frac{1}{2}} \qquad \frac{1}{e} = e^{-1}$$

the power rule for logarithms

The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b(M)$$

How To…

Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm.

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 3 Expanding a Logarithm with Powers

Expand $\log_2(x^5)$.

Solution The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2(x)$$

Try It #3

Expand $\ln(x^2)$.

Example 4 Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Solution Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

Try It #4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Example 5 Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Try It #5

Rewrite $2\log_3(4)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b(y)\\ &= \log_b(6) + \log_b(x) - \log_b(y)\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1})\\ &= \log_b(A) + \log_b(C^{-1})\\ &= \log_b(A) + (-1)\log_b(C)\\ &= \log_b(A) - \log_b(C)\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product. With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 6 **Expanding Logarithms Using Product, Quotient, and Power Rules**

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Solution First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4 y}{7}\right) = \ln(x^4 y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4 y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Try It #6

Expand $\log\left(\frac{x^2 y^3}{z^4}\right)$.

Example 7 **Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression**

Expand $\log(\sqrt{x})$.

Solution

$$\begin{aligned}\log(\sqrt{x}) &= \log(x)^{\frac{1}{2}}\\ &= \frac{1}{2}\log(x)\end{aligned}$$

Try It #7

Expand $\ln(\sqrt[3]{x^2})$.

Q & A…

Can we expand $\ln(x^2 + y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 8 **Expanding Complex Logarithmic Expressions**

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6(64) + \log_6(x^3) + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Quotient Rule.}\\ &= \log_6(2^6) + \log_6(x^3) + \log_6(4x+1) - \log_6(2x-1) && \text{Simplify by writing 64 as } 2^6.\\ &= 6\log_6(2) + 3\log_6(x) + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Power Rule.}\end{aligned}$$

Try It #8

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{x^2-9}\right)$.

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How To...

Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 9 Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Solution Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Try It #9

Condense $\log(3) - \log(4) + \log(5) - \log(6)$.

Example 10 Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x - 1) - 3\log_2((x + 3)^2)$.

Solution We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x - 1) - 3\log_2((x + 3)^2) = \log_2(x^2) + \log_2\left(\sqrt{x - 1}\right) - \log_2((x + 3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2\left(\sqrt{x - 1}\right) - \log_2((x + 3)^6) = \log_2\left(x^2\sqrt{x - 1}\right) - \log_2((x + 3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2\left(x^2\sqrt{x - 1}\right) - \log_2((x + 3)^6) = \log_2\left(\frac{x^2\sqrt{x - 1}}{(x + 3)^6}\right)$$

Try It #10

Rewrite $\log(5) + 0.5\log(x) - \log(7x - 1) + 3\log(x - 1)$ as a single logarithm.

Example 11 Rewriting as a Single Logarithm

Rewrite $2\log(x) - 4\log(x + 5) + \frac{1}{x}\log(3x + 5)$ as a single logarithm.

Solution We apply the power rule first:

$$2\log(x) - 4\log(x + 5) + \frac{1}{x}\log(3x + 5) = \log(x^2) - \log((x + 5)^4) + \log\left((3x + 5)^{x^{-1}}\right)$$

Next we apply the product rule to the sum:

$$\log(x^2) - \log((x + 5)^4) + \log\left((3x + 5)^{x^{-1}}\right) = \log(x^2) - \log\left((x + 5)^4(3x + 5)^{x^{-1}}\right)$$

Finally, we apply the quotient rule to the difference:

$$\log(x^2) - \log\left((x + 5)^4(3x + 5)^{x^{-1}}\right) = \log\left(\frac{x^2}{(x + 5)^4(3x + 5)^{x^{-1}}}\right)$$

Try It #11

Condense $4\left(3\log(x) + \log(x + 5) - \log(2x + 3)\right)$.

Example 12 **Applying of the Laws of Logs**

Recall that, in chemistry, $\text{pH} = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$\text{pH} = -\log(2C)$$

Using the product rule of logs

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

Try It #12

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$, we show

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}$$

Let $y = \log_b(M)$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\log_n(b^y) = \log_n(M) \qquad \text{Apply the one-to-one property.}$$

$$y\log_n(b) = \log_n(M) \qquad \text{Apply the power rule for logarithms.}$$

$$y = \frac{\log_n(M)}{\log_n(b)} \qquad \text{Isolate } y.$$

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)} \qquad \text{Substitute for } y.$$

For example, to evaluate $\log_5(36)$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5(36) = \frac{\log(36)}{\log(5)} \qquad \text{Apply the change of base formula using base 10.}$$

$$\approx 2.2266 \qquad \text{Use a calculator to evaluate to 4 decimal places.}$$

the change-of-base formula

The **change-of-base formula** can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b(M) = \frac{\ln(M)}{\ln(b)} \quad \text{and} \quad \log_b(M) = \frac{\log_n(M)}{\log_n(b)}$$

How To...

Given a logarithm with the form $\log_b(M)$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$.

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   - **a.** The numerator of the quotient will be a logarithm with base n and argument M.
   - **b.** The denominator of the quotient will be a logarithm with base n and argument b.

Example 13 Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5(3)$ to a quotient of natural logarithms.

Solution Because we will be expressing $\log_5(3)$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b(M) = \frac{\ln(M)}{\ln(b)} \qquad \log_5(3) = \frac{\ln(3)}{\ln(5)}$$

Try It #13

Change $\log_{0.5}(8)$ to a quotient of natural logarithms.

Q & A...

Can we change common logarithms to natural logarithms?

Yes. Remember that $\log(9)$ means $\log_{10}(9)$. So, $\log(9) = \frac{\ln(9)}{\ln(10)}$.

Example 14 Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2(10) = \frac{\ln(10)}{\ln(2)} \qquad \text{Apply the change of base formula using base } e.$$

$$\approx 3.3219 \qquad \text{Use a calculator to evaluate to 4 decimal places.}$$

Try It #14

Evaluate $\log_5(100)$ using the change-of-base formula.

Access this online resource for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms (http://openstaxcollege.org/l/proplog)
- Expand Logarithmic Expressions (http://openstaxcollege.org/l/expandlog)
- Evaluate a Natural Logarithmic Expression (http://openstaxcollege.org/l/evaluatelog)

6.5 SECTION EXERCISES

VERBAL

1. How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?

2. What does the change-of-base formula do? Why is it useful when using a calculator?

ALGEBRAIC

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3. $\log_b(7x \cdot 2y)$

4. $\ln(3ab \cdot 5c)$

5. $\log_b\left(\frac{13}{17}\right)$

6. $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7. $\ln\left(\frac{1}{4^k}\right)$

8. $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9. $\ln(7) + \ln(x) + \ln(y)$

10. $\log_3(2) + \log_3(a) + \log_3(11) + \log_3(b)$

11. $\log_b(28) - \log_b(7)$

12. $\ln(a) - \ln(d) - \ln(c)$

13. $-\log_b\left(\frac{1}{7}\right)$

14. $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

16. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17. $\log\left(\sqrt{x^3y^{-4}}\right)$

18. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20. $\log(2x^4) + \log(3x^5)$

21. $\ln(6x^9) - \ln(3x^2)$

22. $2\log(x) + 3\log(x+1)$

23. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$

24. $4\log_7(c) + \frac{\log_7(a)}{3} + \frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25. $\log_7(15)$ to base e

26. $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

27. $\log_{11}(5)$

28. $\log_6(55)$

29. $\log_{11}\left(\frac{6}{11}\right)$

NUMERIC

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30. $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

31. $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$

32. $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33. $\log_3(22)$

34. $\log_8(65)$

35. $\log_6(5.38)$

36. $\log_4\left(\frac{15}{2}\right)$

37. $\log_{\frac{1}{2}}(4.7)$

EXTENSIONS

38. Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6) + \log_{12}(x+2) = 2$. Show the steps for solving.

39. Use the quotient rule for logarithms to find all x values such that $\log_6(x+2) - \log_6(x-3) = 1$. Show the steps for solving.

40. Can the power property of logarithms be derived from the power property of exponents using the equation $b^x = m$? If not, explain why. If so, show the derivation.

41. Prove that $\log_b(n) = \frac{1}{\log_n(b)}$ for any positive integers $b > 1$ and $n > 1$.

42. Does $\log_{81}(2401) = \log_3(7)$? Verify the claim algebraically.