4

Exponential and Logarithmic Functions

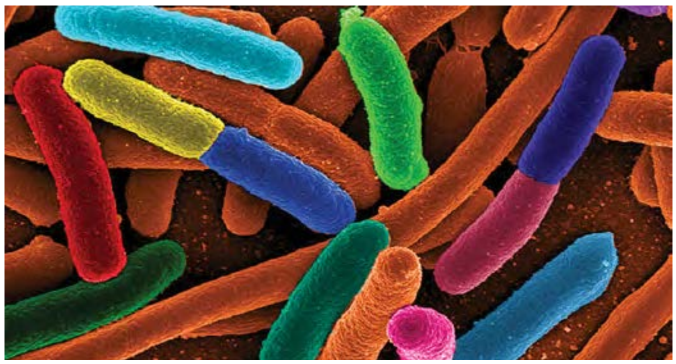

Figure 1 Electron micrograph of *E. Coli* bacteria (credit: "Mattosaurus," Wikimedia Commons)

CHAPTER OUTLINE

Introduction

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years.[16]

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. **Table 1** shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

Table 1

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

16. Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.

LEARNING OBJECTIVES

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

4.3 LOGARITHMIC FUNCTIONS

Figure 1 Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[19]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[20] like those shown in **Figure 1**. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[21] whereas the Japanese earthquake registered a 9.0.[22]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10{,}000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in **Figure 2**, to better estimate the solution.

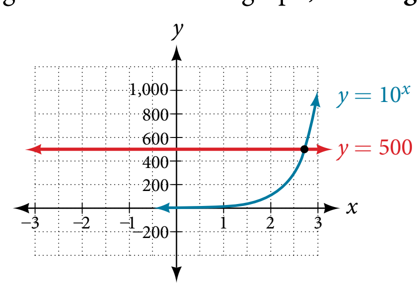

Figure 2

19 http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary. Accessed 3/4/2013.
20 http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc001xgp/#summary. Accessed 3/4/2013.
21 http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.
22 http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc001xgp/#details. Accessed 3/4/2013.

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in **Figure 2** passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x,\ b > 0,\ b \neq 1$$

Note that the base b is always positive.

$=$

$\log_b(x) = y$

to

Think
b to the $y = x$

Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

$=$

$\log_b(c) = a$ means $b^a = c$

to

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

definition of the logarithmic function

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, "the logarithm with base b of x" or the "log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q & A...

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

How To...

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b(x)$ and identify b, y, and x.
2. Rewrite $\log_b(x) = y$ as $b^y = x$.

Example 1 **Converting from Logarithmic Form to Exponential Form**

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ **b.** $\log_3(9) = 2$

Solution First, identify the values of b, y, and x. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$.

b. $\log_3(9) = 2$ Here, $b = 3$, $y = 2$, and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Try It #1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1{,}000{,}000) = 6$ **b.** $\log_5(25) = 2$

Converting From Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Example 2 **Converting from Exponential Form to Logarithmic Form**

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$ **b.** $5^2 = 25$ **c.** $10^{-4} = \frac{1}{10{,}000}$

Solution First, identify the values of b, y, and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$ Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$ Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10{,}000}$ Here, $b = 10$, $x = -4$, and $y = \frac{1}{10{,}000}$. Therefore, the equation $10^{-4} = \frac{1}{10{,}000}$ is equivalent to $\log_{10}\left(\frac{1}{10{,}000}\right) = -4$.

Try It #20

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$ **b.** $5^3 = 125$ **c.** $2^{-1} = \frac{1}{2}$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(8)$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2(8) = 3$.

Now consider solving $\log_7(49)$ and $\log_3(27)$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7(49) = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3(27) = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}}\left(\frac{4}{9}\right)$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2$.

How To...

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of $b : b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x?"

Example 3 Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know $4^3 = 64$ therefore, $\log_4(64) = 3$.

Try It #3

Solve $y = \log_{121}(11)$ without using a calculator.

Example 4 Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Try It #4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

definition of the common logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How To…

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example 5 Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1{,}000)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $10^y = 1{,}000$. Next, we ask, "To what exponent must 10 be raised in order to get 1,000?" We know $10^3 = 1{,}000$ therefore, $\log(1{,}000) = 3$.

Try It #5

Evaluate $y = \log(1{,}000{,}000)$.

How To…

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x, followed by [ )].
3. Press **[ENTER]**.

Example 6 Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by [ )].
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis *Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:*

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Try It #6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 7 Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \qquad \text{Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by [ )].
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Try It #7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln(1) = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

definition of the natural logarithm

A **natural logarithm** is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e = x$ for $x > 0$.

How To...

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x, followed by [ )].
3. Press **[ENTER]**.

Example 8 Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500, followed by [ )].
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Try It #8

Evaluate $\ln(-500)$.

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms (http://openstaxcollege.org/l/intrologarithms)

4.3 SECTION EXERCISES

VERBAL

1. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b(x) = y$ for $b > 0, b \neq 1$.

2. How is the logarithmic function $f(x) = \log_b(x)$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3. How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

4. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

ALGEBRAIC

For the following exercises, rewrite each equation in exponential form.

6. $\log_4(q) = m$

7. $\log_a(b) = c$

8. $\log_{16}(y) = x$

9. $\log_x(64) = y$

10. $\log_y(x) = -11$

11. $\log_{15}(a) = b$

12. $\log_y(137) = x$

13. $\log_{13}(142) = a$

14. $\log(v) = t$

15. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

16. $4^x = y$

17. $c^d = k$

18. $m^{-7} = n$

19. $19^x = y$

20. $x^{-\frac{10}{13}} = y$

21. $n^4 = 103$

22. $\left(\frac{7}{5}\right)^m = n$

23. $y^x = \frac{39}{100}$

24. $10^a = b$

25. $e^k = h$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26. $\log_3(x) = 2$

27. $\log_2(x) = -3$

28. $\log_5(x) = 2$

29. $\log_3(x) = 3$

30. $\log_2(x) = 6$

31. $\log_9(x) = \frac{1}{2}$

32. $\log_{18}(x) = 2$

33. $\log_6(x) = -3$

34. $\log(x) = 3$

35. $\ln(x) = 2$

For the following exercises, use the definition of common and natural logarithms to simplify.

36. $\log(100^8)$

37. $10^{\log(32)}$

38. $2\log(0.0001)$

39. $e^{\ln(1.06)}$

40. $\ln(e^{-5.03})$

41. $e^{\ln(10.125)} + 4$

NUMERIC

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42. $\log_3\left(\frac{1}{27}\right)$

43. $\log_6(\sqrt{6})$

44. $\log_2\left(\frac{1}{8}\right) + 4$

45. $6\log_8(4)$

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46. $\log(10,000)$

47. $\log(0.001)$

48. $\log(1) + 7$

49. $2\log(100^{-3})$

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50. $\ln\left(e^{\frac{1}{3}}\right)$

51. $\ln(1)$

52. $\ln(e^{-0.225}) - 3$

53. $25\ln\left(e^{\frac{2}{5}}\right)$

TECHNOLOGY

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54. $\log(0.04)$

55. $\ln(15)$

56. $\ln\left(\frac{4}{5}\right)$

57. $\log\left(\sqrt{2}\right)$

58. $\ln\left(\sqrt{2}\right)$

EXTENSIONS

59. Is $x = 0$ in the domain of the function $f(x) = \log(x)$? If so, what is the value of the function when $x = 0$? Verify the result.

60. Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

61. Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

62. Is the following true: $\dfrac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$? Verify the result.

63. Is the following true: $\dfrac{\ln(e^{1.725})}{\ln(1)} = 1.725$? Verify the result.

REAL-WORLD APPLICATIONS

64. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65. Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

66. The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula

$$\log\frac{I_1}{I_2} = M_1 - M_2$$

where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0.[23] How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

23 http://earthquake.usgs.gov/earthquakes/world/historical.php. Accessed 3/4/2014.