

Figure 9.1 Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: "intographics"/Pixabay)

Chapter Outline

- 9.1 Solve Quadratic Equations Using the Square Root Property
- 9.2 Solve Quadratic Equations by Completing the Square
- 9.3 Solve Quadratic Equations Using the Quadratic Formula
- 9.4 Solve Quadratic Equations in Quadratic Form
- 9.5 Solve Applications of Quadratic Equations
- 9.6 Graph Quadratic Functions Using Properties
- 9.7 Graph Quadratic Functions Using Transformations
- 9.8 Solve Quadratic Inequalities

🖉 Introduction

Blink your eyes. You've taken a photo. That's what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

⁹¹ Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- > Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- > Solve quadratic equations of the form $a(x h)^2 = k$ using the Square Root Property

Be Prepared!

Before you get started, take this readiness quiz.

- 1. Simplify: $\sqrt{128}$. If you missed this problem, review **Example 8.13**.
- 2. Simplify: $\sqrt{\frac{32}{5}}$.

If you missed this problem, review **Example 8.50**.

3. Factor: $9x^2 - 12x + 4$. If you missed this problem, review **Example 6.23**.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2 . We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the diffe ence of squares.	(x-3)(x+3) = 0
Use the Zero Product Property.	x - 3 = 0 $x - 3 = 0$
Solve each equation.	x = 3 $x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, x = 4, x = -4 and x = 5, x = -5.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If
$$n^2 = m$$
, then *n* is a square root of *m*.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of *k*. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

 $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm \sqrt{k}$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm \sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

Use the Square Root Property.

$$x^{2} = 9$$

$$x = \pm \sqrt{9}$$

$$x = \pm 3$$

So $x = 3$ or $x = -3$.

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

 $x^2 = 7$ Use the Square Root Property. $x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

EXAMPLE 9.1 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

✓ Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2. Use Square Root Property.	Remember to write the ± symbol.	$x = \pm \sqrt{50}$
Step 3. Simplify the radical.	Rewrite to show two solutions.	$x = \pm \sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2}, x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^{2} - 50 = 0$ $(5\sqrt{2})^{2} - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^{2} - 50 = 0$ $(-5\sqrt{2})^{2} - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

> **TRY IT ::** 9.1 Solve: $x^2 - 48 = 0$.

TRY IT :: 9.2 Solve: $y^2 - 27 = 0$.

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

HOW TO :: SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.

- Step 2. Use Square Root Property.
- Step 3. Simplify the radical.
- Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

EXAMPLE 9.2

>

Solve: $3z^2 = 108$.

⊘ Solution

		$3z^2 = 108$
The quadratic Divide by 3 to	term is isolated. make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.		$z^2 = 36$
Use the Squa	re Root Property.	$z = \pm \sqrt{36}$
Simplify the r	adical.	$z = \pm 6$
Rewrite to she	ow two solutions.	z = 6, z = -6
Check the sol	utions:	
$3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $108 = 108 \checkmark$	$3z^2 = 108$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $108 = 108 \checkmark$	

>	TRY IT : : 9.3	Solve: $2x^2 = 98$.
>	TRY IT : : 9.4	Solve: $5m^2 = 80$.

The Square Root Property states 'If $x^2 = k$,' What will happen if k < 0? This will be the case in the next example.

EXAMPLE 9.3

Solve: $x^2 + 72 = 0$.

⊘ Solution

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm \sqrt{-72}$
Simplify using complex numbers.	$x = \pm \sqrt{72} i$
Simplify the radical.	$x = \pm 6\sqrt{2} i$
Rewrite to show two solutions.	$x = 6\sqrt{2}i, x = -6\sqrt{2}i$

Check the solutions:

$x^{2} + 72 = 0$	$x^{2} + 72 = 0$	
$(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$	$(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$	
$6^2(\sqrt{2})^2i^2 + 72 \stackrel{?}{=} 0$	$(-6)^2(\sqrt{2})^2i^2 + 72 \stackrel{?}{=} 0$	
36 • 2 • (−1) + 72 ≟ 0	36 • 2 • (−1) + 72 ² = 0	
0 = 0 ✓	0 = 0 ✓	

>	TRY IT : : 9.5	Solve: $c^2 + 12 = 0$.
>	TRY IT : : 9.6	Solve: $q^2 + 24 = 0$.

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

EXAMPLE 9.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

✓ Solution

		$\frac{2}{3}u^2 + 5$	= 17
Isolate the quadrat	tic term.	$\frac{2}{3}u^{2} =$	= 12
Multiply by $\frac{3}{2}$ to m	nake the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 =$	= <mark>3</mark> •12
Simplify.		U ² =	= 18
Use the Square Roo	ot Property.	<i>u</i> =	$=\pm\sqrt{18}$
Simplify the radical	l.	<i>u</i> =	=±√9•2
Simplify.		<i>u</i> =	$=\pm 3\sqrt{2}$
Rewrite to show tw	o solutions.	$u = 3\sqrt{2},$	$u = -3\sqrt{2}$
Check:			
$\frac{2}{3}u^2 + 5 = 17$	$\frac{2}{3}u^2 + 5 = 17$		
$\frac{2}{3}(3\sqrt{2})^2 + 5 \stackrel{?}{=} 17$	$\frac{2}{3}(-3\sqrt{2})^2 + 5 \stackrel{?}{=} 17$		
$\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$	$\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$		
12 + 5 ≟ 17	12 + 5 ≟ 17		
17 = 17 ✓	17 = 17 ✓		

> **TRY IT ::** 9.7 Solve: $\frac{1}{2}x^2 + 4 = 24$.

> **TRY IT ::** 9.8 Solve:
$$\frac{3}{4}y^2 - 3 = 18$$
.

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

EXAMPLE 9.5

Solve: $2x^2 - 8 = 41$.

⊘ Solution

	$2x^2 - 8 = 41$
Isolate the quadratic term.	$2x^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2} = \frac{49}{2}$
Simplify.	$x^2 = \frac{49}{2}$
Use the Square Root Property.	$x = \pm \sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm \frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm \frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$
Simplify.	$x = \pm \frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x = \frac{7\sqrt{2}}{2}, x = -\frac{7\sqrt{2}}{2}$
Check:	

We leave the check for you.

> **TRY IT ::** 9.9 Solve: $5r^2 - 2 = 34$.

TRY IT :: 9.10 Solve: $3t^2 + 6 = 70$.

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with (x - h).

 $ax^2 = k$ $a(x - h)^2 = k$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x - h)^2$.

EXAMPLE 9.6

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Solve: $4(y-7)^2 = 48$.

⊘ Solution

		$4(y-7)^2 = 48$
Divide both sides by t	he coefficient 4.	$(y-7)^2 = 12$
Use the Square Root	Property on the binomial	$y - 7 = \pm \sqrt{12}$
Simplify the radical.		$y - 7 = \pm 2\sqrt{3}$
Solve for <i>y</i> .		$y = 7 \pm 2\sqrt{3}$
Rewrite to show two	solutions.	$y = 7 + 2\sqrt{3}, y = 7 - 2\sqrt{3}$
Check:		
$4(y-7)^2 = 48$	$4(y-7)^2 = 48$	
$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48$	$4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$	
$4(2\sqrt{3})^2 \stackrel{?}{=} 48$	$4(-2\sqrt{3})^2 \stackrel{?}{=} 48$	
4(12) ² / ₌ 48	4(12) ≟ 48	
48 = 48 ✓	48 = 48 ✓	

> **TRY IT ::** 9.11 Solve: $3(a-3)^2 = 54$.

> **TRY IT ::** 9.12 Solve: $2(b+2)^2 = 80$.

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

EXAMPLE 9.7

Solve: $(x - \frac{1}{3})^2 = \frac{5}{9}$.

✓ Solution

	$\left(x-\frac{1}{3}\right)^2$	=	<u>5</u> 9
Use the Square Root Property.	$x-\frac{1}{3}$	=	$\pm \sqrt{\frac{5}{9}}$
Rewrite the radical as a fraction of square roots.	$x-\frac{1}{3}$	=	$\pm \frac{\sqrt{5}}{\sqrt{9}}$
Simplify the radical.	$x-\frac{1}{3}$	=	$\pm \frac{\sqrt{5}}{3}$
Solve for <i>x</i> .	x	=	$\frac{1}{3} \pm \frac{\sqrt{5}}{3}$
Rewrite to show two solutions.	$x = \frac{1}{3} + \frac{\sqrt{5}}{3},$	x	$=\frac{1}{3}-\frac{\sqrt{5}}{3}$

Check: We leave the check for you.

> **TRY IT ::** 9.13
Solve:
$$\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$$
.
> **TRY IT ::** 9.14
Solve: $\left(y + \frac{3}{4}\right)^2 = \frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

EXAMPLE 9.8

Solve: $2(x-2)^2 + 3 = 57$.

✓ Solution

	$2(x-2)^2 + 3$	=	57
Subtract 3 from both sides to isolate the binomial term	$2(x-2)^2$	=	54
Divide both sides by 2.	$(x-2)^2$	=	27
Use the Square Root Property.	x-2	=	$\pm \sqrt{27}$
Simplify the radical.	x-2	=	$\pm 3\sqrt{3}$
Solve for <i>x</i> .	x	=	$2 \pm 3\sqrt{3}$
Rewrite to show two solutions.	$x = 2 + 3\sqrt{3},$	<i>x</i> =	$2-3\sqrt{3}$
Check:			
We leave the check for you.			

> **TRY IT ::** 9.15 Solve: $5(a-5)^2 + 4 = 104$.

TRY IT :: 9.16 Solve: $3(b+3)^2 - 8 = 88$.

Sometimes the solutions are complex numbers.

EXAMPLE 9.9

>

Solve: $(2x - 3)^2 = -12$.

⊘ Solution

	$(2x-3)^2 = -12$
Use the Square Root Property.	$2x - 3 = \pm \sqrt{-12}$
Simplify the radical.	$2x - 3 = \pm 2\sqrt{3} i$
Add 3 to both sides.	$2x = 3 \pm 2\sqrt{3} i$
Divide both sides by 2.	$x = \frac{3 \pm 2\sqrt{3}i}{2}$
Rewrite in standard form.	$3, 2\sqrt{3}i$
Simplify.	$x = \frac{1}{2} \pm \frac{1}{2}$
	$x = \frac{3}{2} \pm \sqrt{3} i$
Rewrite to show two solutions.	$x = \frac{3}{2} + \sqrt{3}i, x = \frac{3}{2} - \sqrt{3}i$
Check:	
We leave the check for you.	



> **TRY IT ::** 9.18 Solve: $(2t - 8)^2 = -10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x - h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

EXAMPLE 9.10

Solve: $4n^2 + 4n + 1 = 16$.

⊘ Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n+1)^2 = 16$
Use the Square Root Property.	$2n+1 = \pm \sqrt{16}$
Simplify the radical.	$2n+1 = \pm 4$
Solve for <i>n</i> .	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1 \pm 4}{2}$ $n = \frac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2}$, $n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2}, \qquad n = -\frac{5}{2}$

Check:

$4n^2 + 4n + 1 = 16$	$4n^2 + 4n + 1 = 16$	
$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$	$4\left(-\frac{5}{2}\right)^{2}+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$	
$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16$	$4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$	
9 + 6 + 1 ≟ 16	25 – 10 + 1 ≟ 16	
16 = 16 ✓	16 = 16 ✓	

> **TRY IT ::** 9.19 Solve: $9m^2 - 12m + 4 = 25$. > **TRY IT ::** 9.20 Solve: $16n^2 + 40n + 25 = 4$.

MEDIA : :

►

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)
- Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)

9.1 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$	2 . $b^2 = 144$	3. $r^2 - 24 = 0$
4. $t^2 - 75 = 0$	5. $u^2 - 300 = 0$	6. $v^2 - 80 = 0$
7. $4m^2 = 36$	8. $3n^2 = 48$	9. $\frac{4}{3}x^2 = 48$
10. $\frac{5}{3}y^2 = 60$	11. $x^2 + 25 = 0$	12. $y^2 + 64 = 0$
13. $x^2 + 63 = 0$	14. $y^2 + 45 = 0$	15. $\frac{4}{3}x^2 + 2 = 110$
16. $\frac{2}{3}y^2 - 8 = -2$	17 . $\frac{2}{5}a^2 + 3 = 11$	18 . $\frac{3}{2}b^2 - 7 = 41$
19. $7p^2 + 10 = 26$	20. $2q^2 + 5 = 30$	21. $5y^2 - 7 = 25$
22. $3x^2 - 8 = 46$		

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property In the following exercises, solve each equation.

23. $(u-6)^2 = 64$	24. $(v + 10)^2 = 121$	25. $(m-6)^2 = 20$
26. $(n+5)^2 = 32$	27. $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$	28. $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$
29. $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$	30. $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$	31. $(a-7)^2 + 5 = 55$
32. $(b-1)^2 - 9 = 39$	33. $4(x+3)^2 - 5 = 27$	34. $5(x+3)^2 - 7 = 68$
35. $(5c+1)^2 = -27$	36. $(8d-6)^2 = -24$	37. $(4x - 3)^2 + 11 = -17$
38. $(2y+1)^2 - 5 = -23$	39. $m^2 - 4m + 4 = 8$	40 . $n^2 + 8n + 16 = 27$
41. $x^2 - 6x + 9 = 12$	42. $y^2 + 12y + 36 = 32$	43. $25x^2 - 30x + 9 = 36$
44. $9y^2 + 12y + 4 = 9$	45. $36x^2 - 24x + 4 = 81$	46. $64x^2 + 144x + 81 = 25$

Mixed Practice

In the following exercises, solve using the Square Root Property. **47.** $2r^2 = 32$ **48.** $4t^2 = 16$ **50.** $(b+7)^2 = 8$ **49.** $(a-4)^2 = 28$ **51.** $9w^2 - 24w + 16 = 1$ **52.** $4z^2 + 4z + 1 = 49$ **53.** $a^2 - 18 = 0$ **54.** $b^2 - 108 = 0$ **55.** $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$ **56.** $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$ **57**. $m^2 + 12 = 0$ **58.** $n^2 + 48 = 0$. **59.** $u^2 - 14u + 49 = 72$ **60.** $v^2 + 18v + 81 = 50$ **61.** $(m-4)^2 + 3 = 15$ **62.** $(n-7)^2 - 8 = 64$ **63.** $(x+5)^2 = 4$ **64.** $(v-4)^2 = 64$ **65.** $6c^2 + 4 = 29$ **66.** $2d^2 - 4 = 77$ **67.** $(x-6)^2 + 7 = 3$ **68.** $(v - 4)^2 + 10 = 9$

Writing Exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x + 2)^2 = 16$.

Self Check

^(a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x - h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

b If most of your checks were:

...confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

...with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

...no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be

overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.