Chapter 5

Linear Programming: A Geometric Approach[1]

5.1 Chapter Overview

In this chapter, you will learn to:

1. Solve linear programming problems that maximize the objective function.
2. Solve linear programming problems that minimize the objective function.

5.2 Maximization Applications

Application problems in business, economics, and social and life sciences often ask us to make decisions on the basis of certain conditions. These conditions or constraints often take the form of inequalities. In this section, we will look at such problems.

A typical **linear programming** problem consists of finding an extreme value of a linear function subject to certain constraints. We are either trying to maximize or minimize our function. That is why these linear programming problems are classified as **maximization** or **minimization problems**, or just **optimization problems**. The function we are trying to optimize is called an **objective function**, and the conditions that must be satisfied are called **constraints**. In this chapter, we will do problems that involve only two variables, and therefore, can be solved by graphing. We begin by solving a maximization problem.

Example 5.1

Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

We start by choosing our variables.

Let $x =$ The number of hours per week Niki will work at Job I.

and $y =$ The number of hours per week Niki will work at Job II.

[1]This content is available online at <http://cnx.org/content/m18903/1.2/>.

Example 5.5
Professor Symons wishes to employ two students, John and Mary, to grade the homework papers for his classes. John can mark 20 papers per hour and charges \$5 per hour, and Mary can mark 30 papers per hour and charges \$8 per hour. Each student must be employed at least one hour a week to justify their employment. If Mr. Symons has at least 110 homework papers to be marked each week, how many hours per week should he employ each student to minimize his cost?

Solution
We choose the variables as follows:
Let $x =$ The number of hours per week John is employed.
and $y =$ The number of hours per week Mary is employed.
The objective function is

$$C = 5x + 8y \tag{5.16}$$

The fact that each student must work at least one hour each week results in the following two constraints:

$$x \geq 1 \tag{5.17}$$

$$y \geq 1 \tag{5.18}$$

Since John can grade 20 papers per hour and Mary 30 papers per hour, and there are at least 110 papers to be graded per week, we get

$$20x + 30y \geq 110 \tag{5.19}$$

The fact that x and y are non-negative, we get
$x \geq 0$, and $y \geq 0$.
The problem has been formulated as follows.
Minimize $C = 5x + 8y$
Subject to: $x \geq 1$

$$y \geq 1 \tag{5.20}$$

$$20x + 30y \geq 110 \tag{5.21}$$

$$x \geq 0; y \geq 0 \tag{5.22}$$

To solve the problem, we graph the constraints as follows:

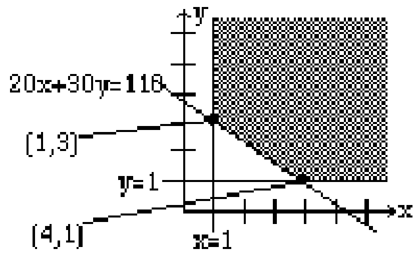

Figure 5.5

Again, we have shaded the feasibility region, where all constraints are satisfied.

Since the extreme value of the objective function always takes place at the vertices of the feasibility region, we identify the two critical points, (1, 3) and (4, 1). To minimize cost, we will substitute these points in the objective function to see which point gives us the minimum cost each week. The results are listed below.

Critical Points	Income
(1,3)	$5\,(1) + 8\,(3) = \$29$
(4,1)	$5\,(4) + 8\,(1) = \$28$

Table 5.3

The point (4, 1) gives the least cost, and that cost is \$28. Therefore, we conclude that Professor Symons should employ John 4 hours a week, and Mary 1 hour a week at a cost of \$28 per week.

Example 5.6

Professor Hamer is on a low cholesterol diet. During lunch at the college cafeteria, he always chooses between two meals, Pasta or Tofu. The table below lists the amount of protein, carbohydrates, and vitamins each meal provides along with the amount of cholesterol he is trying to minimize. Mr. Hamer needs at least 200 grams of protein, 960 grams of carbohydrates, and 40 grams of vitamins for lunch each month. Over this time period, how many days should he have the Pasta meal, and how many days the Tofu meal so that he gets the adequate amount of protein, carbohydrates, and vitamins and at the same time minimizes his cholesterol intake?

	Pasta	Tofu
Protein	8g	16g
Carbohydrates	60g	40g
Vitamin C	2g	2g
Cholesterol	60mg	50mg

Table 5.4

Solution

We choose the variables as follows.

Let x = The number of days Mr. Hamer eats Pasta.

and y = The number of days Mr. Hamer eats Tofu.

Since he is trying to minimize his cholesterol intake, our objective function represents the total amount of cholesterol C provided by both meals.

$$C = 60x + 50y \tag{5.23}$$

The constraint associated with the total amount of protein provided by both meals is as follows:

$$8x + 16y \geq 200 \tag{5.24}$$

Similarly, the two constraints associated with the total amount of carbohydrates and vitamins are obtained, and they are

$$60x + 40y \geq 960 \tag{5.25}$$

$$2x + 2y \geq 40 \tag{5.26}$$

The constraints that state that x and y are non-negative are

$$x \geq 0, \text{ and } y \geq 0 \tag{5.27}$$

We summarize all information as follows:

Minimize $C = 60x + 50y$

Subject to: $8x + 16y \geq 200$

$$60x + 40y \geq 960 \tag{5.28}$$

$$2x + 2y \geq 40 \tag{5.29}$$

$$x \geq 0; y \geq 0 \tag{5.30}$$

To solve the problem, we graph the constraints as follows.

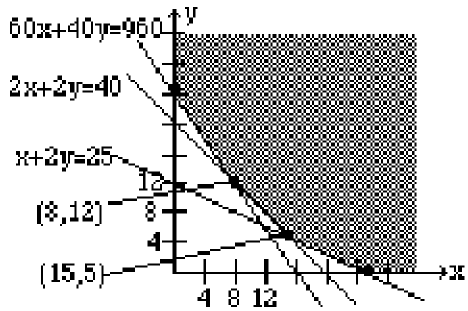

Figure 5.6

Again, we have shaded the unbounded feasibility region, where all constraints are satisfied.

To minimize the objective function, we find the vertices of the feasibility region. These vertices are (0, 24), (8, 12), (15, 5) and (25, 0). To minimize cholesterol, we will substitute these points in the objective function to see which point gives us the smallest value. The results are listed below.

Critical Points	Income
(0,24)	$60\,(0) + 50\,(24) = 1200$
(8,12)	$60\,(8) + 50\,(12) = 1080$
(15,5)	$60\,(15) + 50\,(5) = 1150$
(25,0)	$60\,(25) + 50\,(0) = 1500$

Table 5.5

The point (8, 12) gives the least cholesterol, which is 1080 mg. This states that for every 20 meals, Professor Hamer should eat Pasta 8 days, and Tofu 12 days.

Although the method of solving minimization problems is similar to that of the maximization problems, we still feel that we should summarize the steps involved.

5.7: Minimization Linear Programming Problems

1. Write the objective function.
2. Write the constraints.
    a) For standard minimization linear programming problems, constraints are of the form: $\mathrm{ax} + \mathrm{by} \geq c$
    b) Since the variables are non-negative, include the constraints: $x \geq 0$; $y \geq 0$.
3. Graph the constraints.
4. Shade the feasibility region.
5. Find the corner points.
6. Determine the corner point that gives the minimum value.

a) This can be done by finding the value of the objective function at each corner point.
b) This can also be done by moving the line associated with the objective function.
c) There is the possibility that the problem has no solution.